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PROBABILITY AND WINNING LOTTERY

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ABSTRACT

A lottery is a form of gambling that involves the drawing of numbers for a prize. The aim of this paper is to explain the role of math rules in lottery and to calculate the chances of winning. In a simple 6-from-49 lottery Which a player chooses six numbers from 1 to 49 (no duplicates are allowed), If all six numbers on the player's ticket match those produced in the official drawing then the player is a jackpot winner. By a quick review of math we will show that the chance of winning is 1 in 13,983,816. The chances of winning are reduced even more by increasing the group from which numbers are drawn. Findings of this study reveal that a basic understanding of statistics and mathematics is enough to understand that lottery players are often losers.
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Keywords: *lottery, gambling, mathematics, probability.*

INTRODUCTION

It is impossible to deny the role of mathematics rules on numerous aspects of daily life [1]. We can't succeed in any job, science and even entertainment without possessing good knowledge of mathematics. One of the recreations used by many people is betting with lottery cards. Millions of people each year spend a lot of money on lottery in the hope of being richer [2].There are more than 70 % of UK population (about 45 million people) and 57% of the American population, (about 181 million people) buys at least one lottery ticket in a year [3].We need to use math to understand the mathematics behind lottery and avoid losing [4]. People who gamble have usually incorrect common beliefs and attitudes about luck and prediction. They think that they will usually win even when they are aware of the odds [6]. Studies show that the most lottery players are found to have less education than non-players and also they do not know the real mechanism and rule of lottery which it is not actually clear in the first look [5]. By a rule of thumb, we will understand that betting on lottery is not demanded when we notice the mathematical view of it [3]. The aim of this study is to explain the role of math rules in lottery as a very common form of gambling which can give us a better view of it.

PROBABILITY RULES

The probability rules in mathematics are used to explain the chances of winning in lottery. Probability is an estimate of the chance of winning divided by the total number of chances available. The probability of event tells us how likely it is that the event will occur and it can be expressed as an ordinary fraction, a percentage or as a proportion between 0 and 1.For example if there are 10 numbers and a player owns two of them, the probability of winning is 2 in 10 or 2/10 or 20% or p=0.20. If you know the probability of an event occurring, it is easy to compute the probability that the event does not occur. If P(A) is the probability of Event A, then 1 - P(A) is the probability that the event does not occur so The probability of loosing is 8 in 10 or 8/10 or 80% or p=0.80. A random event is very likely to happen if its probability is close to 1 and it is not likely to happen if the probability is close to 0 [6].

RESULTS AND DISCUSSION

Lottery is a form of gambling which each player chooses k numbers from 1-n; if the k numbers match the numbers drawn by the lottery, the player is a jackpot winner. In a common 6/49 game; each player chooses six diverse numbers from a range of 1-49. If the six chosen numbers match the numbers drawn by the lottery, the ticket holder is a jackpot winner. In order to find the probability of winning lottery we will have to figure out “the number of winning lottery numbers” and “the total number of possible lottery numbers” [6].

The first number drawn has a 1 in 49 chance of matching. When the draw comes to the second number, there are now only 48 numbers left (because the numbers already drawn) so there is now a 1 in 48 chance of predicting this number. So for each of the 49 ways of choosing the first number there are 48 different ways of choosing the second. This means that the probability of correctly predicting 2 numbers drawn from 49 in the correct order is calculated as 1 in 49×48. On drawing the third number there are only 47 ways of choosing the number; but of course we could have arrived at this point in any of 49×48 ways, so the chances of correctly predicting 3 numbers drawn from 49, again in the correct order, is 1 in $49 \times 48 \times 47$. This continues until the sixth number has been drawn. Eventually we have 49 × 48 × 47 × 46 × 45 × 44 ways of choosing 6 number from 1-49, which can also be written as 49! / (49-6)! , or 49 factorial divided by 43 factorial. This equals to 10,068,347,520 ways of choosing. However, the order of the 6 numbers is not significant. It means , if a ticket has the numbers 1, 2, 3, 4, 5, and 6, it wins as long as all the numbers 1 through 6 are drawn, no matter what order they come out in. there are 6 × 5 × 4 × 3 × 2 × 1 = 6! = 720 orders in which they could be drawn. Dividing 10,068,347,520 by 720 gives us 13,983,816 as the total number of possible 6/49 lottery tickets. In order to win the large jackpot in the lottery you must hold the ticket matches all 6 numbers so the probability of the large jackpot in the 6/49 lottery is 1 / 13,983,816 [7].

Despite of the low chance of winning in lottery, Most of the lottery gamblers misunderstand “the number of drawn number” and “the number of lottery numbers on their ticket”. They feel it is good if they have a wide choice of choosing numbers. For example they think they have more chance in 7/50 than a 6/49 lottery because they can choose more numbers however with a simple math calculation we will understand that they are wrong [8].

The chance of winning in 7/50 lottery is calculated exactly like 6/49 lottery but with a difference in the drawn numbers and the numbers on the players ticket. In the 7/50 lottery we have 50 × 49 × 48 × 47 × 46 × 45 × 44 ways of choosing 7 number from 1-50 which can also be written as 50!/(50-7)! , or 50 factorial divided by 43 factorial which it is 503,417,276,000 ways of choosing. Dividing 503,417,276,000 by 5040 Gives us 99,884,400 as the total number of possible 7/50 lottery tickets. In order to win the large jackpot you must hold the ticket matches all 7 numbers so the probability of the large jackpot in the 7/50 lottery is 1 / 99,884,400 which it is much less than 1 / 13,983,816 (probability of winning 6/49 lottery) [7].

Statistics shows that in the year 2003, there were 49 % of American people betting on lottery cards and spending about 19.9 billion dollars on it [9].It is hard to believe that this huge amount of money was spent in gambling and there are just a few winners. It is obvious that people have to trust mathematic rules to avoid losing money in gambling.

CONCLUSION

A basic understanding of statistics and mathematics is enough to understand that lottery gamblers are often losers.

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MATHEMATICS AND PARADOXES: A RELATION OF LOVE AND HATE: MONTY HALL PARADOX

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ABSTRACT

The paradox concept is closely connected with the evolution of mathematics. Most often it gives rise to the creation of new notions and theories, while it leads to mathematical interpretations of the issue of which is related. Some of the most famous paradoxes consist of Zeno's, Galileo's and Russell's, which are closely related to the concept of infinity in mathematics. On the other hand, the existing mathematical theory is sufficient to interpret paradoxical situations, which are contrary to common sense, such as Monty Hall paradox. In this paper we describe the paradoxes associated with the infinity while in addition we will focus on the Monty Hall paradox, showing that mathematics and paradoxes have a relation of love and hate.

INTRODUCTION

At the current work we examine various paradoxes. To be precise, we are interested in Zeno's, Galileo's, Russell's and Monty Hall paradox. The first two have to do with the notion of mathematical infinity in the following sense: Zeno's paradox, specifically Achilles and the tortoise, refers to the infinitude of a finite object while Galileo's paradox treats the infinity as a number by stating that the cardinality of the set of even numbers and the set of natural numbers is the same. In the framework of the Mathematical Logic Russell's paradox points out the difficulty of defining the notion of set and proving that the already existing definition of Cantor was wrong. Finally, the Monty Hall paradox using probability theory shows that the existing mathematical theory is sufficient to interpret paradoxical situations, which are contrary to common sense.

The outline of the current paper will be the following: we start by discussing briefly what paradox is and the notion of mathematical infinity as it is studied from the ancient years to twentieth century. Then we give some historical aspects of Zeno, Galileo and Russell and proceed to the full description of each paradox as well as the formal explanation of each one.

WHAT IS A MATHEMATICAL PARADOX?

A **paradox** is a statement that, despite apparently sound reasoning from true premises, leads to a self-contradictory or a logically unacceptable conclusion. A paradox involves contradictory yet interrelated elements that exist simultaneously and persist over time.

Mathematical paradoxes are statements that run counter to one's intuition, sometimes in simple, playful ways, and sometimes in extremely esoteric and profound ways. It should perhaps come as no surprise that a field with as rich a history as mathematics should have many of them. They range from very simple, everyday common-sense issues, to advanced ones at the frontiers of mathematics. Most often mathematical paradox gives rise to the creation of new notions and theories, while it leads to mathematical interpretations of the issue of which is related.

Mathematical paradoxes carry a special challenge, a hidden message, thinking of which provides the learner with an opportunity to refine their incomplete understanding.

MATHEMATICAL INFINITY AND A BRIEF HISTORY ABOUT IT

Infinity (symbol: ∞, John Wallis 1657) is an abstract concept describing something *without any bound* or *larger than any number*. The idea of infinity has been a source of interest, fascination, and occasionally frustration for people. Infinity came up not just in mathematics, but also in science, art, and religion.

David Hilbert has said about infinity: ‘No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite’.

Infinity is not a simple concept. Unless a care is exercised, especially when we are dealing with infinity as if it was a common number, paradoxes arise readily.

Let’s see a brief history of Mathematical infinity which shows the importance of it.

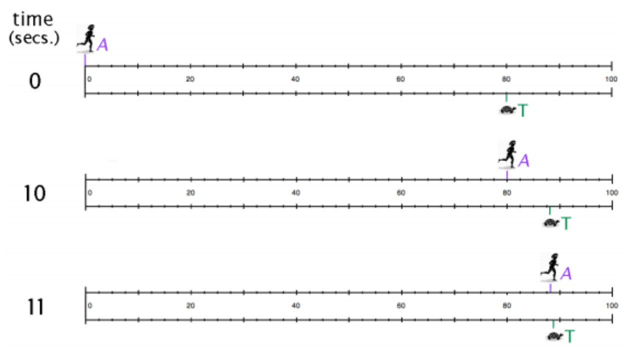
The ancient Greeks expressed infinity by the word *apeiron*, which had connotations unbounded, indefinite, undefined, and formless. One of the earliest appearances of infinity in mathematics is attributed to Pythagoras (*c.* 580–

500 BCE) and his followers initially believed that any aspect of the world could be expressed by an arrangement involving just the whole numbers, but they were surprised to discover that the diagonal and the side of a square are incommensurable. Plato's (428/427–348/347 BCE) philosophical principle is the 'Infinity and Finite', which is the definition of Being. Aristotle (384–322 BCE) influenced subsequent thought for more than a millennium with his rejection of "actual" infinity, which he distinguished from the "potential" infinity of being able to count without end. Theaetetus of Athens (417–368 BC) studied the irrational numbers as square roots [Euclid's Elements, book X]). Eudoxus of Cnidus (400–350 BCE) and Archimedes (285–212/211 BCE) developed a technique, later known as the method of exhaustion, in order to avoid the use of actual infinity [Euclid's Elements, book V].

The issue of infinitely small numbers led to the discovery of calculus in the late 1600s by the English mathematician Isaac Newton and the German mathematician Gottfried Wilhelm Leibniz. Newton introduced his own theory of infinitely small numbers, or infinitesimals, to justify the calculation of derivatives, or slopes. The use of infinitesimal numbers finally gained a firm footing with the development of nonstandard analysis by the German-born mathematician Abraham Robinson in the 1960s. A more direct use of infinity in mathematics arises with efforts to compare the sizes of infinite sets, such as the set of points on a line (real numbers) or the set of counting numbers. In the early 1600s, the Italian scientist Galileo Galilei addressed this and a similar non intuitive result now known as Galileo's paradox. The German mathematician Richard Dedekind in 1872 suggested a definition of an infinite set as one that could be put in a one-to-one relationship with some proper subset. The confusion about infinite numbers was resolved by the German mathematician Georg Cantor beginning in 1873, who demonstrated that the set of rational numbers is the same size as the counting numbers; hence, they are called countable, or denumerable. Along with a principle known as the axiom of choice, the proof method of Cantor's theorem can be used to ensure an endless sequence of transfinite cardinals continuing past $\aleph_1$ to such numbers as $\aleph_2$ and $\aleph_{\aleph_0}$. The continuum problem is the question of which of the alephs is equal to the continuum cardinality. Cantor conjectured that $c = \aleph_1$; this is known as Cantor's continuum hypothesis (CH). In the early 1900s a thorough theory of infinite sets was developed. This theory is known as ZFC, which stands for Zermelo – Fraenkel set theory with the axiom of choice. CH is known to be undecidable on the basis of the axioms in ZFC. In 1940 the Austrian-born logician Kurt Gödel was able to show that ZFC cannot disprove CH, and in 1963 the American mathematician Paul Cohen showed that ZFC cannot prove CH. Recent work suggests that CH may be false and that the true size of c may be the larger infinity $\aleph_2$.

ZENO’S PARADOX: ACHILLES AND THE TORTOISETHE CONTRADICTORY INFINITE AND FINITE

Before we present and analyze Zeno’s paradox it is worth to say a few words about Zeno.
Zeno of Elea (490–430 BC) was a pre-Socratic Greek philosopher of Magna Graecia, member of the Parmenides’ Eleatic School. Little is known for certain about Zeno's life. Primary source of his biographical information is Plato's *Parmenides* and Aristotle's *Physics*. Aristotle called him the inventor of the dialectic.
Zeno's paradoxes have puzzled, challenged, influenced, inspired, infuriated, and amused philosophers, mathematicians, and physicists for over two millennia. The most famous are the arguments against motion described by Aristotle in his *Physics:* Achilles and the tortoise, the dichotomy, the arrow and the moving rows.
In this work we concentrate to the paradox of Achilles and the tortoise, so we start by typing the paradox.
Say the race is over a distance of 100 meters, and for simplicity, assume that each contestant moves at a constant speed.
The tortoise goes 0.8 meters per second. Achilles is 10 times as fast, i.e., he goes 8 meters per second! So we’d better give the tortoise a huge head start. Let’s make it 80 meters, 4/5 of the total distance. Namely, we have:
After 10 sec: Achilles will have run 80m (staring point of the tortoise) and tortoise has moved 8m.
After 1 sec: Achilles will have run 8m (second point of tortoise) and tortoise has moved 0.8 m.
After 0,1 sec: Achilles will have run 0,8m (third point of tortoise) and tortoise has moved 0.08 m.
And so on and so on.



Thus, whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. Achilles **must reach infinitely many points** where the tortoise has already been before he catches up.
So, as Aristotle said***, he can never overtake the tortoise***!
This is obviously wrong, but why?
An initial explanation is:
The total time it would take Achilles to catch up, in seconds, is
10 + 1 + 0.1 + 0.01 + 0.001 +… .
This is an ***infinite*** series, infinitely many numbers are being added up.
But does that mean the total is infinite?
Actually: **10 + 1 + 0.1 + 0.01 + 0.001 +… = 11.111…,** but, **that's not infinite**!
In fact, we have 11<11,111...<12.
But what is it exactly?
We know 0.333333… = 1/3 then dividing both sides by 3 gives 0.111111… = 1/9.
So, Achilles needs just 11 and 1/9sec to catch up.
The flaw in Zeno's argument is his unstated assumption that the sum of an infinite series (or at least an infinite series like this, where every term is greater than zero) cannot be ***finite***. But the situation wasn't really clear until Newton and Liebniz invented *calculus* in the late 17th century and the calculus's theory give us the following formal explanation:

$$\sum_{n=0}^{\infty} 10\left(\frac{1}{10}\right)^n = 10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots + \frac{1}{1000\ldots0} + \cdots = 11\frac{1}{9}$$

In mathematical terminology:

1. 12 is an upper bound for the sum of this infinite series
2. the sum of an **infinite** series, in which each term is $\frac{1}{10}$ the preceding one, is **finite**.

In fact,

- this series (like our case) is convergent, i.e., it has a **finite** sum
- the argument that 12 is an upper bound for its sum can easily be made into a formal proof
- if each term of a series is the previous term times a constant ratio (as in our problem), it's an **infinite geometric series**.

But which **infinite** geometric series converges?
All those (and only those) in which the ratio of consecutive terms is > –1 and < +1,
so that the absolute values of the terms get progressively smaller.
Thus, generally, we can prove that:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r},\ \ for\ |r| < 1.$$

GALILEO'S PARADOX: IS THE WHOLE GREATER THAN THE PART?

Who is Galileo Galilei? He was born in 1564 in Pisa, Italy. He began studying to become a priest, switched to pursue a medical degree, and left that incomplete to become immersed in mathematics. In Padua, he studied astronomy and this led to developing the Copernican theory that the earth revolves around the sun. In 1611, the Catholic Church condemned "*On the Revolution of the Heavenly Orbs*" by Copernicus. Galileo was warned to avoid defending the theories of Copernicus. In 1633 was condemned and placed under house arrest. Galileo died in 1642.
In his scientific work, the "Two New Sciences", the famous Italian scientist wrote about the contradiction with perfect squares, namely,
there are positive integer numbers 1,2,3,4,5... and the numbers 1, 4, 9, 16, 25 ... which are their perfect squares.
The question is: is the cardinality of the set of Natural Numbers larger than the cardinality of the set of even numbers?
At a first look the answer would be Yes, since the set of even numbers is subset of the set of Natural Numbers. However, Galileo suspected that the cardinalities of the two sets are equal but he could not prove this contradiction.
Later Dedekind and Cantor resolved the contradiction by defining **one-to-one correspondence of infinite sets**, that is

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \dots n \dots\dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 4 & 9 & 16 & 25 & \dots n^2 \dots\dots\dots \end{array}$$

The idea behind the one-to-one correspondence is that for the sets with infinite cardinality a part of the set could be just as large as the whole, while with finite sets, a part is always smaller than the whole.
Thus, *often it looks as a paradox, but from the mathematical point of view there is no paradox*
So, Euclid's Principle (Common Notion 5: The whole is greater than the part [i.e., strictly greater than any proper part]) applies to finite sets, but not το infinite sets.

RUSSELL'S LOGICAL PARADOX – THE SET OF ALL SETS

Sir Bertrand Arthur William Russell (18 May 1872–2 February 1970) was British philosopher, logician, mathematician, historian, writer, political activist and anti-imperialism. He was cofounder of analytic philosophy with Gottlob Frege, G. E. Moore, Ludwig Wittgenstein and one of the 20th century's premier logicians. He wrote ***Principia Mathematica*** with A. N. Whitehead. His philosophical essay

"***On Denoting***" has been considered a "paradigm of philosophy. He went to prison for his pacifism during World War I.
In 1950 Russell was awarded the ***Nobel Prize in Literature.***
Mathematician Georg Cantor and the other early set theorists were operating in a world of what we now call "***naive set theory***". Their intuitive idea of what a set is was very loosely defined, that is:
Definition: ***a set is a collection of things***.
Bertrand Russell showed in 1901 that some attempted formalizations of ***the naive set theory led to a contradiction***.
To be precise, let A be the set of all sets which do not contain themselves so, $A=\{S \mid S \notin S\}$.
Bertrand Russell stated the following question: Is $A \in A$?
Suppose $A \in A$, then by definition of A, $\mathbf{A \notin A}$.
Suppose $A \notin A$, then by definition of A, $\mathbf{A \in A}$.
This is the known Russell's Paradox, which showed that Cantor's definition is wrong and led to build up an axiomatic theory. In 1908 two ways of avoiding the paradox were proposed. The first one is the Russell's type theory and the second was Zermelo axiomatic set theory which is evolved into the now-canonical Zermelo-fraenkel set theory (ZFC) and where the axioms went well beyond Gottlob Frege's axioms of extensionality and unlimited set abstraction.

MONTY HALL PARADOX CONTRADICT TO COMMON SENSE

Let's start with the history of Monty Hall paradox. Firstly, the **Monty Hall problem** is a brain teaser, in the form of a probability puzzle (Gruber, Krauss and others) which based on the American television game show *Let's Make a Deal* and named after its original host, Monty Hall.
The problem was originally posed in a letter by Steve Selvin to the *American Statistician* in 1975 (Selvin 1975a, b). The game originated in the United States in 1963 and has since been produced in many countries throughout the world. The program was created and produced by Stefan Hatos and Monty Hall, the latter serving as its host for many years. It became famous as a question from a reader's letter quoted in Marilyn vos Savant's "Ask Marilyn" column in *Parade* magazine in 1990 (vos Savant 1990a).
The Monty Hall problem is the following. Suppose you are on a game show, and you are given the choice of three doors. Behind one door is a car; behind the others, goats.
We start by giving the host's strategy and then we analyze the optimal way a player could follow in order to maximize the probability of winning.
The host always plays according to the following rules:

- The host must always open a door that was not picked by the contestant (Mueser and Granberg 1999).
- The host must always open a door to reveal a goat and never the car

- The host must always offer the chance to switch between the originally chosen door and the remaining closed door.

For example, you pick a door, say No. 1, and the host, who knows what it is behind the doors, without opening your chosen door, he opens one of the remaining doors, for example No. 3, which has a goat. He then says to you, "Do you want to switch to door No. 2 instead of No. 1?

The best strategy of the contestant is the following: the player should answer *yes* because if the answer was *no* then obviously the probability of winning is 1/3 (Figure 1), while if the player switches then the probability is 2/3 because car has a 1/3 chance of being behind the player's pick and a 2/3 chance of being behind one of the other two doors.

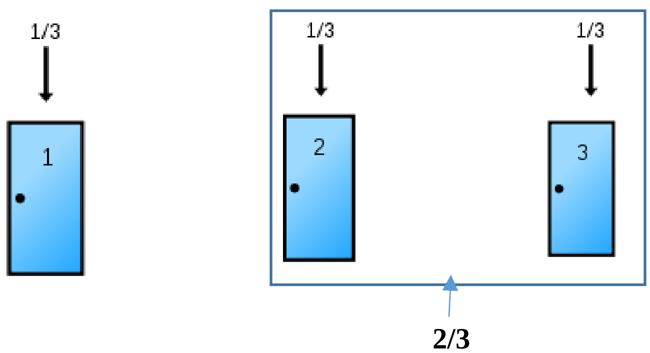


Figure 1

if the answer was *yes* then the odds for the two sets don't change but the odds move to **0** for the open door and **2/3** for the closed door (Figure 2).

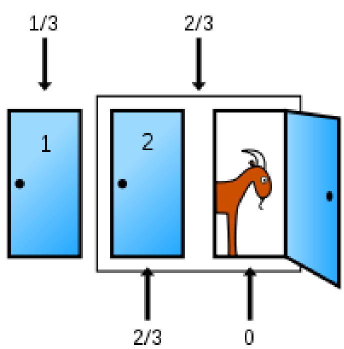


Figure2

The next table summarizes all the possible cases.

Behind door 1	Behind door 2	Behind door 3	Result if staying at door 1	Result if switching to the door offered
Car	Goat	Goat	**Wins car**	Wins goat
Goat	**Car**	Goat	Wins goat	**Wins car**
Goat	Goat	**Car**	Wins goat	**Wins car**

Many readers of vos Savant's column refused to believe switching is beneficial despite her explanation. Especially, approximately 10,000 readers wrote to the magazine including nearly 1,000 with PhDs, most of them claiming vos Savant was wrong (Tierney 1991). Even when given explanations, simulations, and formal mathematical proofs, many people still do not accept that switching is the best strategy (vos Savant 1991a).

Paul Erdős, one of the most prolific mathematicians in history, remained unconvinced until he was shown a computer simulation demonstrating the predicted result (Vazsonyi 1999).

Formalize the Problem by Probabilities we have initially, the odds on door 1, door 2, and door 3 are 1 : 1 : 1.

After the player has chosen door 1, *according to* ***Bayes' rule****,* the posterior odds on the location of the car, given that the host opens door 3, *are equal to the prior odds multiplied by the Bayes factor*. So, the chance that the host opens door 3 is: 50% if the car is behind door1, 100% if the car is behind door2, 0% if the car is behind door3, thus, the ***Bayes*** factor consists of the ratios **1/2 : 1 : 0**. Let's the events *C1*, *C2* and *C3* are: the car is behind respectively door 1, 2, 3, (probability 1/3). The host opening door 3 is described by *H3.*

The player initially chooses door 1(event *X1* and conditional probability *P*(*Ci*|*X1*) = 1/3).

$$P(H3|C1,X1) = \frac{1}{2}, P(H3|C2,X1) = 1,\ P(H3|C3,X1) = 0$$

So,

$$P(C2|H3,X1)=\frac{P(H3|C2,X1)P(C2\cap X1)}{P(H3\cap X1)}=$$
$$\frac{P(H3|C2,X1)P(C2\cap X1)}{P(H3|C1,X1)P(C1\cap X1)+P(H3|C2,X1)P(C2\cap X1)+P(H3|C3,X1)P(C3\cap X1)}$$
$$=\frac{P(H3|C2,X1)P(C2\cap X1)}{P(H3|C1,X1)+P(H3|C2,X1)+P(H3|C3,X1)}=\frac{1}{\frac{1}{2}+1+0}$$
$$=2/3$$

CONCLUSION

- Paradoxes are the fear of mathematical theories
- On the other hand paradoxes review the mathematical theories leading them to new theories free from contradictions, and also point out that common sense poses errors
- So, there is a relation between paradoxes and mathematics a relationship of *love and hate*.

AKNOWLEDGMENTS

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WHY MUSIC AFFECTS US?

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ABSTRACT

Science and music look like two completely different things. Science studies regularities and facts while art is subjective and individual. Science can be found everywhere around us, hence also in music. This project connects mathematics, physics, chemistry and biology with music. Biology explains the physiology of the ear, the way we hear and how colours can be displayed using sound. Physics can help us describe the characteristics of sound, which is the medium of music. Chemistry shows what happens in the brain when you listen to music and why it makes you feel good. And, because all musical compositions have a certain mathematical structure, sets of tones can be connected using the mathematical group theory. All of this can make you listen to music in a completely new way.

Music is an art whose medium is sound. Sound is an oscillation wave of particles and to understand it better, it is necessary to understand its properties. In physics, they are: pitch, volume, duration and colour.

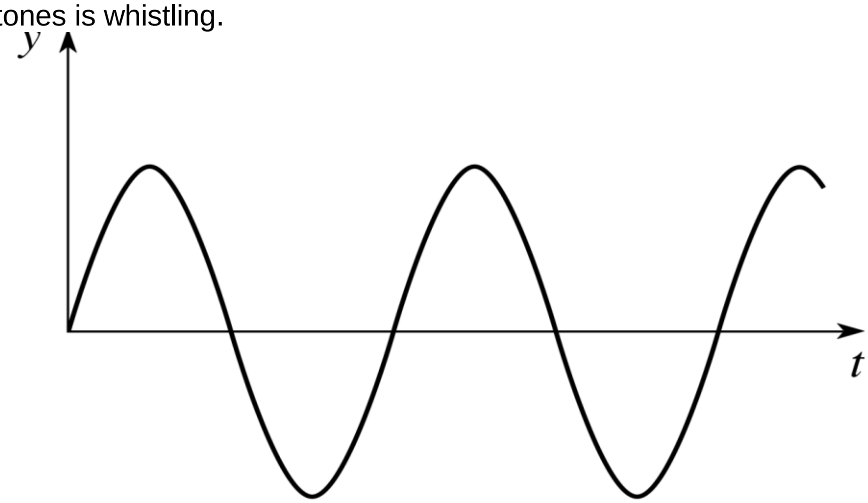
PHYSICAL PROPERTIES OF SOUND

The pitch is a property of sound that allows scaling sounds by their frequencies and it only depends on a sound's frequency; the higher the frequency, the higher the pitch. Due to its connection to frequencies it is measured in hertz (Hz).
Volume, also referred to as loudness, is a measure of the amplitude of a sound wave, a scale which informs you of how loud a sound is. It is measured in decibel (dB).
Duration is the time that sound is perceived by someone.
Colour is the number of aliquot tones that sound over the main tone. The higher the number of aliquot tones, the sound is more pleasant to the ear.

- In physics, there are two types of sound: a pure tone and complex sound.

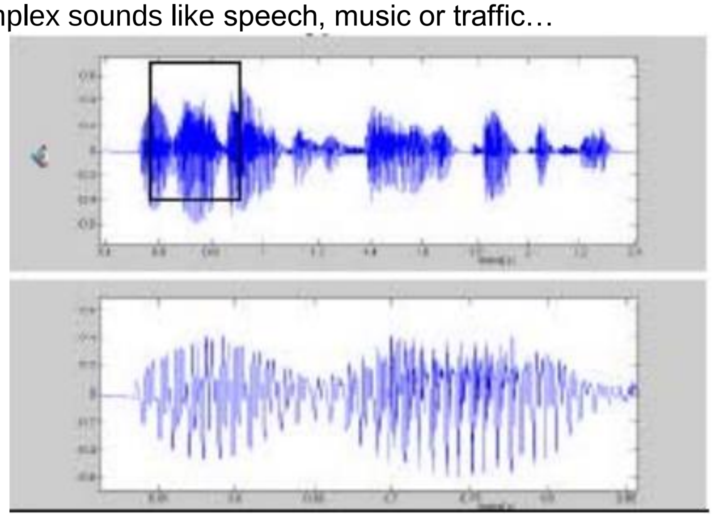
PURE TONE

Pure tones are created when the source of sound is oscillating regularly (pure). This creates a regular sound wave that cannot be broken down. An example of pure tones is whistling.



COMPLEX SOUND

Complex sounds are combinations of pure tones. They can be broken into a series of pure tones. The lowest frequency of a complex sound is called the main frequency, while others are higher harmonics. Most sounds in everyday live are complex sounds like speech, music or traffic…



GROUP THEORY

Mathematical group theory can be used to describe the connections of harmonic triads in music. This way, any two triads can be assigned a numerical value and be linked using a mathematical function. In this case, these numbers have all the properties of mathematical groups and behave according to the principles of group theory.

PROPERTIES OF A GROUP

In the mathematical group theory, a group is defined by certain properties: the function must be bijective, it must be associative and both a neutral and inverse element must exist too.

One-to-one correspondence (bijective function)
This means that a function that maps all the elements of one set to another (where all elements of one set are paired up with exactly one element of the second set) must exist.

$$\boldsymbol{f}: \boldsymbol{S} \rightarrow \boldsymbol{S}'$$

Associativity
All elements of the group must have the property to be grouped in certain ways for any given mathematical operation.

$$(\boldsymbol{a} * \boldsymbol{b}) * \boldsymbol{c} = \boldsymbol{a} * (\boldsymbol{b} * \boldsymbol{c})$$

Neutral element
An element **e** that, given any mathematical operation, will not change the value of another element.

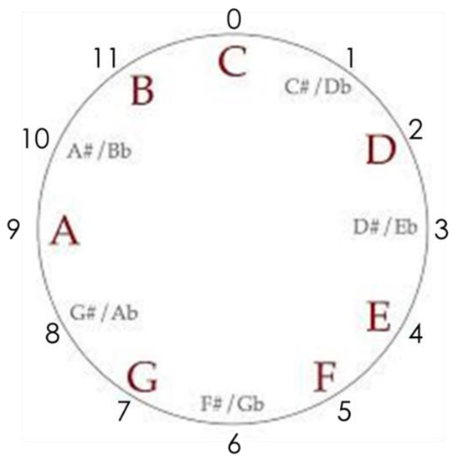
$$\boldsymbol{a} * \boldsymbol{e} = \boldsymbol{a}$$

Inverse element
An element **a** and $\mathbf{a}^{-1}$ must exist, the following equation is true for the element:

$$\boldsymbol{a} * \boldsymbol{a}^{-1} = \boldsymbol{e}$$

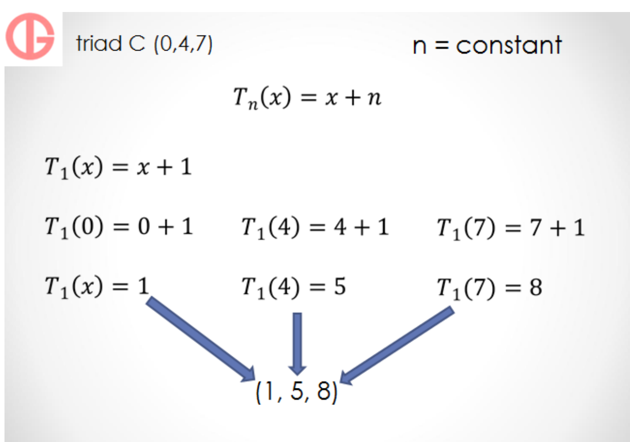
THE MUSICAL CLOCK

It is a graphical display of the numerical values assigned to tones on a musical scale. These numbers are all positive integers from 0 to 11 and it uses the notation $\mathbb{Z}_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$. This is called the set of integers modulo 12.



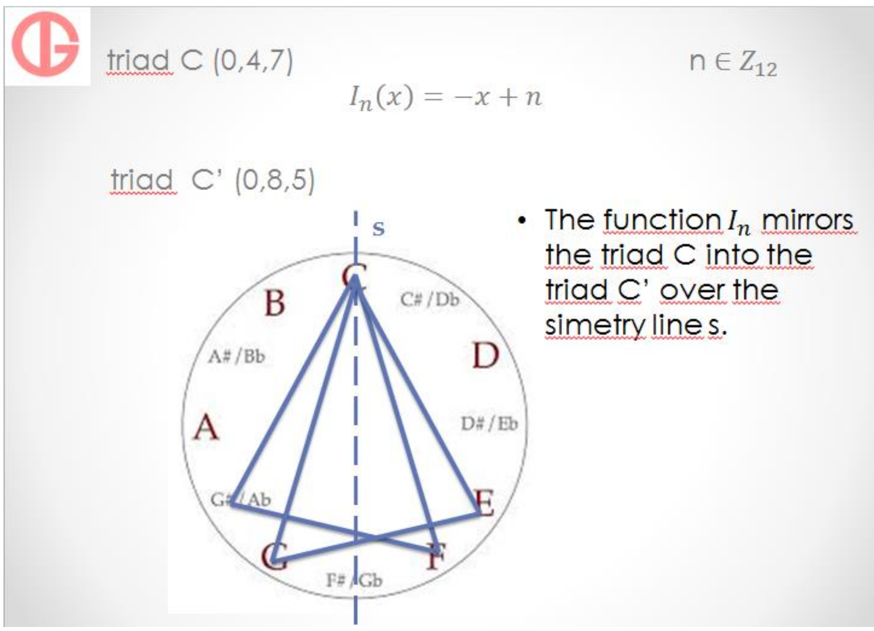
TRANSLATION FUNCTION

The function $\boldsymbol{T_n}: \mathbb{Z}_{\mathbf{12}} \to \mathbb{Z}_{\mathbf{12}}$, defined by the formula $\boldsymbol{T_n(x) = x + n}$ mod 12 is called transposition by n.
This function can output the numerical values of a triad that is harmonic with the triad whose values were used as **x**, where **n** is both a positive integer mod 12 and a constant.



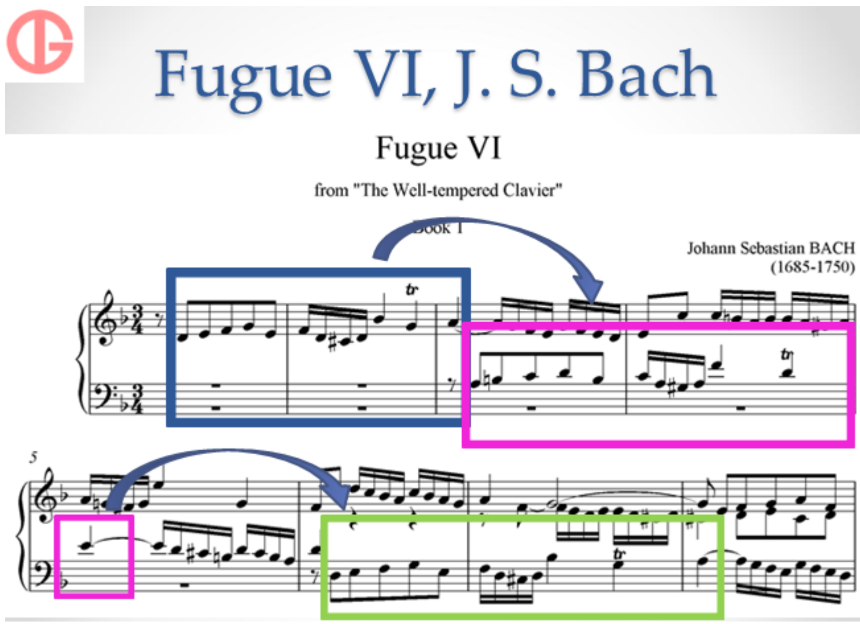
INVERSION FUNCTION

The function $\boldsymbol{I_n}: \mathbb{Z}_{\mathbf{12}} \to \mathbb{Z}_{\mathbf{12}}$, defined by the formula $\mathbf{I_n(x) = -x + n}$ is called inversion about n.
This function can output the numerical values of a triad that is harmonic with the triad whose values were used. It is similar to the translation function in definition, but it is actually very different. It is a function of symmetry, which means that it mirrors the graphical positions of tones on the musical clock over a symmetry line.



- The transpositions and inversions are functions which have inputs and outputs: pitches.

- The use of these functions is practical and can be found in many musical compositions



- This is an example on Johan Sebastian Bach's Fugue VI., where the melody in the rectangles is translated using the T_n function as indicated by the arrows

SONOCHROMATISM

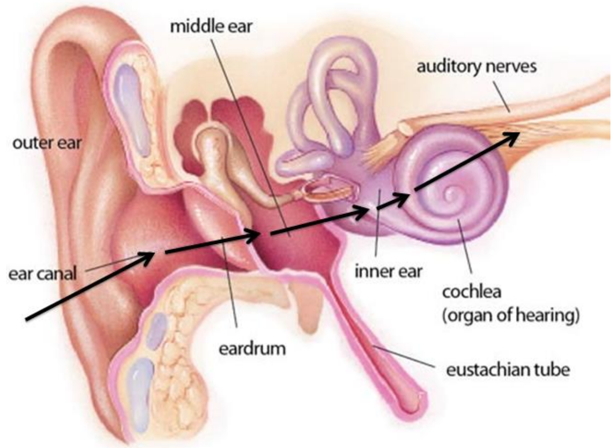
Sonochromatism is a way of perceiving colours trough sound using the sonochromatic scale. Using special devices, colour-blind people can gain the ability to experience colour through sound. The British artist Neil Harbisson used this method to create colourful artworks despite being colour-blind.

PURE SONOCHROMATIC SCALE		
(invisible)	Ultraviolet	Over 717.591 Hz
	Violet	607.542 Hz
	Blue	573.891 Hz
	Cyan	551.154 Hz
	Green	478.394 Hz
	Yellow	462.023 Hz
	Orange	440.195 Hz
	Red	363.797 Hz
(invisible)	Infrared	Below 363.797 Hz

SONOCHROMATIC MUSIC SCALE (basic 12/360)		
	Rose	**E**
	Magenta	**D#**
	Violet	**D**
	Blue	**C#**
	Azure	**C**
	Cyan	**B**
	Spring	**A#**
	Green	**A**
	Chartreuse	**G#**
	Yellow	**G**
	Orange	**F#**
	Red	**F**

HOW DO WE HEAR?

Since sound travels through air as a wave, complex living beings have adapted to be able perceive and react to it. In the case of humans, the sound wave enters the outer ear and reaches the eardrum through the ear canal. The sound vibrates the eardrum and the small bones in the middle ear. Then the vibration travels to the inner ear and the cochlea where, using small sensible hairs, vibrations are turned into electrical impulses. These travel through the auditory nerves and to the brain where these impulses are turned into our perception of sound.



SPIDER WEB AND MUSIC

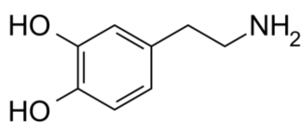
Spider web is truly a miracle of nature and the firmest and most elastic natural creation. It has been used to make violin wires due to its properties. More than 300 treads of spider silk are needed to make one violin. Its sound is described as very pleasant to the ear, silky and soft.

WHY DOES MUSIC AFFECT US?

Listening to music harmonises the left and right side of the brain. It can regulate blood pressure, circulation and berating. It can also enhance memory and encourage positive thinking and creativity, that is why it is recommended to listen to music while studying. One can also become addicted to music.

DOPAMINE

Dopamine is a neurotransmitter in the brain. It is released when we listen to music, have fun, do sports or eat. It is also referred to as “the molecule of happiness”. It is one of the most important molecules in the chemistry of the brain and life without it would be impossible.



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THE BLACK AND WHITE CHALLENGE

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ABSTRACT

In this paper, we discuss the black and white challenge. This problem includes a rotating square that contains four square tiles, each with two different sides, a black and a white. While being blindfolded the player gives orders to a second person who executes the moves and rotates the square after each move. Which are the moves that the player has to do in order to achieve monochrome? Firstly, we present the factors that must be taken into consideration in order to answer the aforementioned question. Afterwards, we solve the problem both intuitively and by utilizing tools from Graph Theory. Lastly, we delve into some extra generalizations that emerged while working on the project.

1. INTRODUCTION

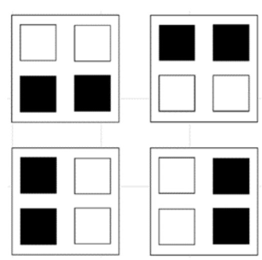
The black and white challenge consists of a rotating square containing four square tiles. Each tile has two sides: a black one and a white one. The goal of the player is to achieve monochrome, which means that all tiles appear with the same colored side facing upwards. The number of people participating in each game are two. One is the blindfolded player, who gives orders to a second person, who executes the moves. An example of the moves that can be executed are: “swipe the down-right tile”, “swipe the down-left tile and the upper-right tile”. These moves must be specific and are given by the blindfolded person who is obviously unaware of the initial configuration of the tiles. The tiles can be placed in various states, specifically 16, which we call situations and are explained in more detail in what follows. It is important to note that there is the possibility of the initial configuration being monochrome. However, this does not signify the immediate completion of the challenge since in order for the player to win he/she should have performed a minimum of one move. Moreover, when the player achieves monochrome he/she is to be informed by the one who executes the moves. Although monochrome can be achieved with various techniques, the purpose of our challenge is to present a solution to the problem that does not depend on the parameter of luck. The goal is to create an algorithm that will always solve the challenge in the minimum number of moves possible, regardless of the initial configuration. At first glance, the black and white challenge seems like a problem unrelated to mathematics. However, during one's encounter with the problem it is realized that even such a problem can be explained with the use of mathematical knowledge.

2. PRELIMINARIES

The black and white challenge can be approached in a variety of ways, but in this paper, we focus on two simple ways of solving the problem: the first is with the use of logic and critical thinking and the second by employing tools of Graph Theory.

We first observed that the different moves that can be executed and the way that the tiles can be placed are specific. As far as the way that the tiles are placed, the different situations are shown in Figure 1.

After we played the game and executed a series of moves, we realized that, since our task is to achieve monochrome, it does not really matter if we make all tiles black or white. Suppose that the initial configuration of the tiles is two black tiles up and the two bottom tiles white. That would be the same with the initial position being the reverse, that is two white tiles up and the bottom ones black since the problem would be solved by swapping either two consecutive tiles (in horizontal position).



Now observe that the square may be rotated. This means that there is symmetry in the sense that there is no difference between the two reverse states just mentioned, they both exist simultaneously! Even better, there is no difference where the two white tiles are, meaning in a horizontal or vertical position, since any move by the player would correspond to the same change of state no matter the orientation of the square: a rotation would render it symmetrically equal.

According to this figuration, we pick a representative from each family of possible situations and eliminate the rest. Thus, our final list of possible situations is depicted in Figure 2.

We proceed with the different moves that can be executed. We realize that the rotation of the square affects not only the situations but also the moves. Namely, the moves may not be extremely specific, such as “move the two upper tiles” or “change the upper left and downright tile” because, since the square rotates randomly, the downright tile may become upper left tile in a single rotation. Therefore, the specific moves lose their property; they are not at all specific but random moves! Thus, there is no reason in executing such moves but performing general moves that are presented in Figure 3. Note that these general moves are only three; There exists no other move.

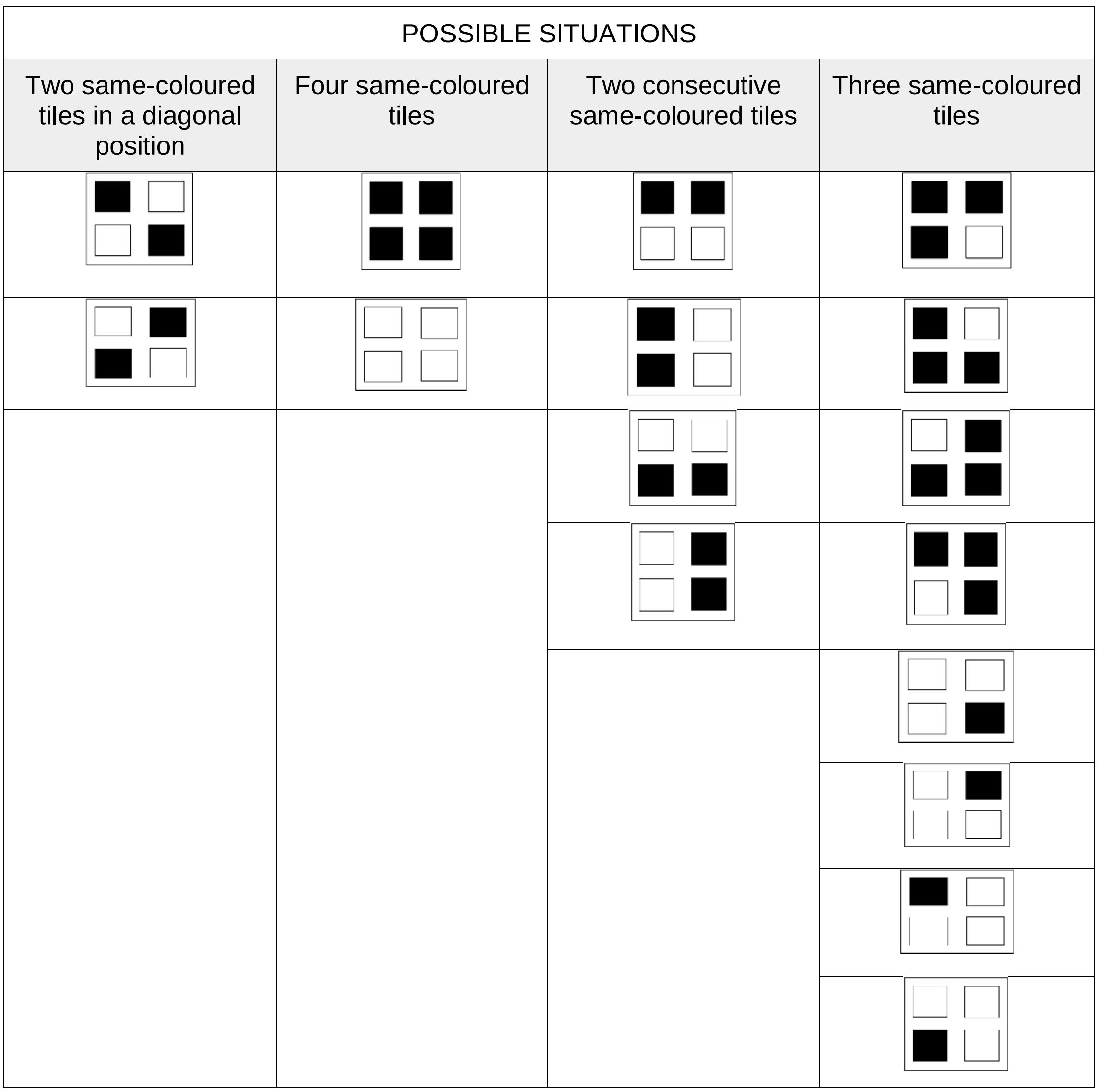


Figure 1: Possible situations for the four tiles

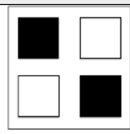
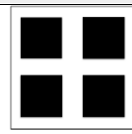
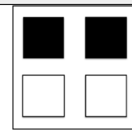
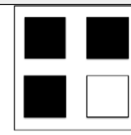
THE FOUR SITUATIONS			
Two same-coloured tiles in a diagonal position (2-2Diagonal)	Four same-coloured tiles (4-0)	Two consecutive same-coloured tiles (2-2Consecutive)	Three same-coloured tiles (3-1)
			

Figure 2: The four representative situations

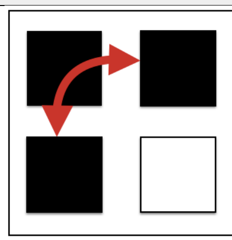
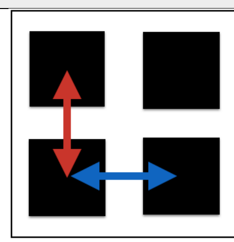
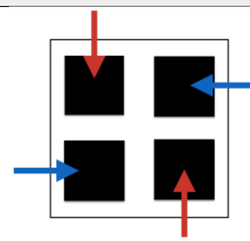
DIFFERENT MOVES		
The change of two diagonal tiles	The change of two consecutive tiles	The change of 1 random tile
		

Figure 3: The different moves

Of course, one may suggest the addition of another move, the change of all four tiles, a move that we have not included in Figure 3. By changing all four tiles, we achieve nothing. Take as an example the situation of two same-colored tiles in a diagonal position as in Figure 4. A change of all four tiles makes no actual difference in the position of the squares. It may appear to be different, but as was explained above and highlighted in Figures 2 and 3, it is the same situation. Similarly, we comprehend how the change of three tiles has the same effect as the change of a single tile.

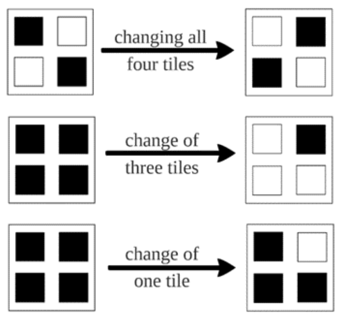


Figure 4: An example

3. SOLUTION WITH THE USE OF LOGIC AND CRITICAL THINKING

Our first thought of solving the problem, after realizing the preliminaries, was to treat each situation as a stand-alone case. The situations are four (see Figure 2) and we tried to solve each of them with the least number of moves. Our hope was to combine all the solutions and formulate the ultimate solution of the black and white challenge.

Initially, we realize that the 2-2Diagonal situation is the easiest case to solve since the only move needed is the change of two diagonal tiles.

Another simple situation to solve is the 4-0 situation. We arrive to the conclusion that two diagonal changes bring the 4-0 situation back to monochrome. In this way, by changing two diagonal tiles we combine the solutions of both situations just mentioned. Thus, our algorithm so far contains two diagonal changes as shown in Figure 5.

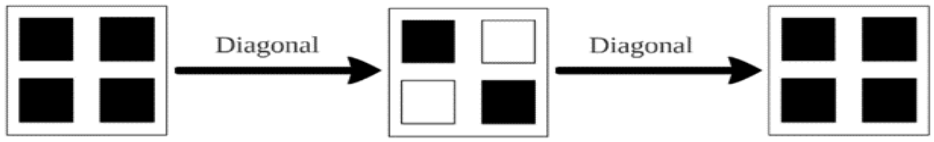


Figure 5: Solution of the first two situations

When trying to solve the 2-2Consecutive situation our goal is to make this situation a 2-2Diagonal situation that we already know how to solve. In this case, the move that helps us is the change of two consecutive tiles. If we are lucky, we achieve monochrome. If not, then the 2-2Consecutive situation has turned into a 2-2Diagonal situation, which we know how to solve (see Figure 5).

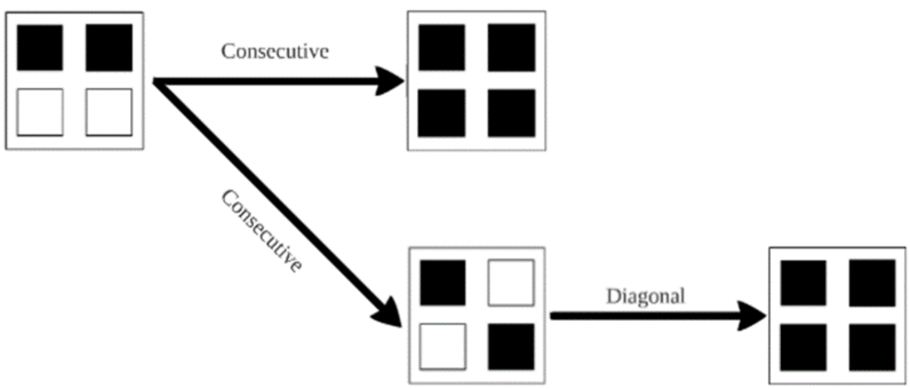


Figure 6: Solution of the 2-2Consecutive situation

Thus, our algorithm so far consists of two diagonal changes, one consecutive and one diagonal change.

Finally, the only situation that remains unsolved is the 3-1 situation. Observe that if the initial configuration is the 3-1 situation, the change of two consecutive or two diagonal tiles has no effect on it. As a result, we conclude that the only move that can alter this situation is the change of one random tile. Again, if we are lucky, we may achieve monochrome at once! However, this may not be the case; we drop to either the 2-2Consecutive or the 2-2Diagonal situation and we realize that if we repeat the procedure outlined above, the problem is solved.

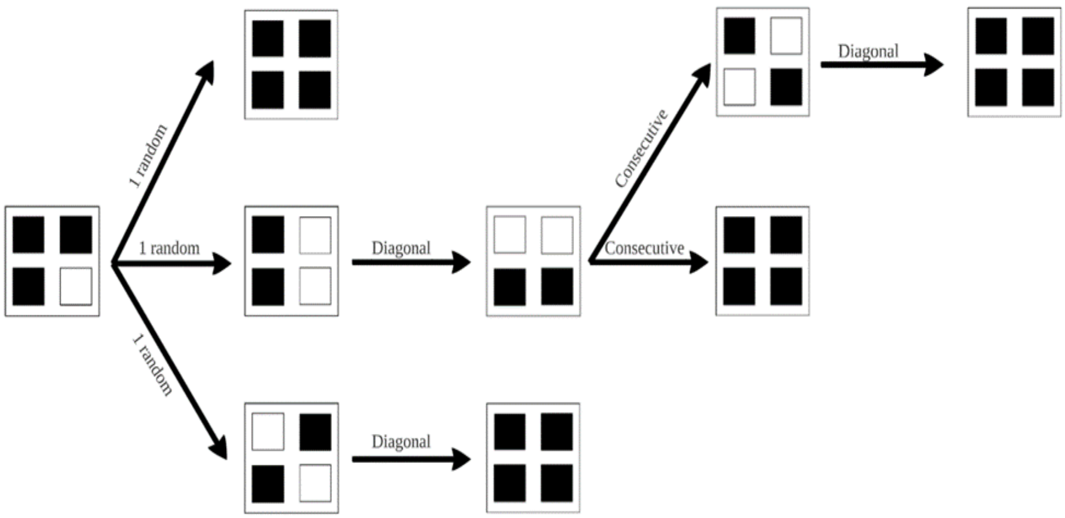


Figure 7: Solution of the (3-1) case

Therefore, the final algorithm for the black and white challenge consists of the eight moves:

1. Diagonal change
2. Diagonal change
3. Concecutive change
4. Diagonal change
5. 1 random tile
6. Diagonal change
7. Consecutive change
8. Diagonal change

4. SOLUTION WITH THE USE OF A GRAPH

The solution of the black and white challenge presented in the previous Section required logic and critical thinking. In this Section, we unravel the hidden mathematics behind this challenge and present an alternative solution using tools from Graph Theory.

A graph is the depiction of information as a collection of points (the vertices) and lines that connect two points (the edges) and a single point (a loop). Therefore, we need to decide what information will be represented by vertices and what by edges. Similar to the way that lines are used to connect each vertex, in our challenge the moves connect the situations. In other words, the execution of a move leads from one situation to another. Hence, in our graph, the edges represent the moves and the vertices represent the situations. Moreover, because we have three different moves we create three different-coloured edges, each one representing a certain move. We perform each move for every situation and discover all the possible outcomes, which means that from each vertex there will be all three different-coloured edges departing from it. We present the resulting graph in Figure 8.

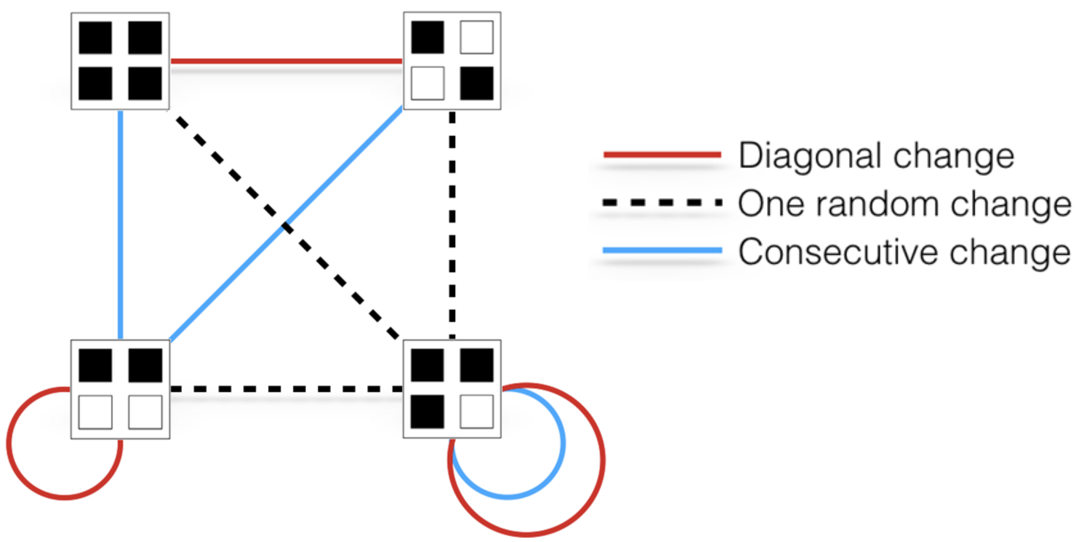


Figure 8: The graph for the black and white challenge

Our immediate observation concerns the existence of the three loops; a loop is the depiction of a move that leaves a certain situation unchanged. We have to take advantage of these loops and tackle initially the situations that do not have loops, because any move will not change the form of the rest. Therefore, firstly we examine the situations 2-2Diagonal and 4-0.

The change of two diagonal tiles (red edge) solves the 2-2Diagonal situation but the 4-0 situation is turned into a 2-2Diagonal situation. Hence, if we repeat a diagonal change we have solved the 4-0 situation too. So, if our first moves are two diagonal changes, the situations 4-0 and 2-2Diagonal have been solved, whereas the remaining situations are unchanged (because of the loops). To continue, we focus on the bottom part of the graph in Figure 8. Even though the situations 2-2Consecutive and 3-1 both have loops, there is a difference in their number; the 2-2Consecutive situation has one less loop than the 3-1 situation. We start with the 2-2Consecutive situation. We realize that we need to perform the change of two consecutive tiles (blue edge). There are two possible outcomes, monochrome and the 2-2Diagonal situation that is solved with the change of two tiles in a diagonal position, as mentioned before. As a result, our algorithm so far consists of moves that solve all situations but the 3-1 situation. Moreover, this situation remains unchanged since all the edges used lead back to this same vertex.

Since the moves we have performed leave the 3-1 situation unchanged, the only move we can do in order to alter the situation is the change of one random tile. The possible outcomes are three: monochrome (if one is lucky), 2-2Diagonal and 2-2Consecutive. What remains is to add the solutions of the 2-2Diagonal and 2-2Consecutive (one diagonal change, one consecutive change, one diagonal change) and therefore we have solved our problem. As we expected, the algorithm is the same and whatever the initial configuration or the state of the player (intelligence or even a robot), one certainly solves the challenge with the use of eight, at most, moves.

5. QUERIES - EXTENSIONS

The study of the black and white challenge led us to some additional questions. Does the parameter of rotation interfere with the final solution? What is the mean value of the number of moves depending on the initial configuration and the rotation of the square? What if the rotating instrument was a pentagon (the number of tiles was five) instead of a square? These queries and extensions are presented in this Section.

5.1 Does the parameter of rotation interfere with the final solution?

In a non-rotating square, the number of moves would be equal or smaller than the number of moves needed to solve the rotating square. This occurs since, theoretically, the square could always return to the same situation as the one before the rotation. On the other hand, the parameter of rotation indicates that there is symmetry in the sense of the situations of this challenge. This is a crucial “hypothesis” that one must use in order to be able to find a valid solution. Thus, as paradoxical as it may sound, the rotating parameter actually simplifies the challenge!

5.2 Calculation of the mean value

The algorithm that we provided at the end of Sections 3 and 4 indicates that the maximum number of moves needed to achieve monochrome is eight. However, it may happen that a player achieves monochrome in a smaller amount of moves, not to mention after the first move if the initial configuration is 2-2Diagonal. In this Section, we calculate the mean value of the number of moves and examine whether we could lower it by devising a (slightly) different algorithm.

First, we examine the mean value of the algorithm as it stands.

If one assumes the worst-case scenario, then the eight (3-1) configurations need 8 moves each, the four (2-2C) configurations need 4 moves each, the two (4-0) need 2 moves and the (2-2D) case is solved in a single move. The mean value is equal to the total amount of moves over the total amount of situations, which gives

$$\frac{8\cdot 8+4\cdot 4+2\cdot 2+2\cdot 1}{16}=5.375$$

Of course, this does not make much sense since we may not face a worst-case scenario. In fact, we have to take into account the probability of each scenario. The (2-2D) case is always solved in one move whereas the (4-0) case in two moves. The (2-2C) case stays the same after the execution of the first two moves and it can be solved in 3 or 4 moves as it is shown in Figure 9. It is easy to see that for this case we have

$$3\cdot\frac{1}{2}+4\cdot\frac{1}{2}=3.5$$

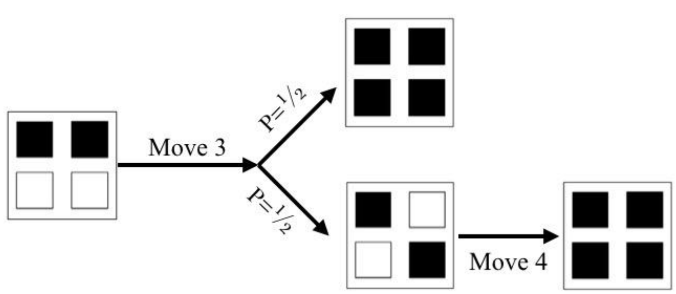


Figure 9: Probabilities for the (2-2C) configuration

Lastly, the (3-1) situation remains unchanged after the first four moves. We summarize the probabilities for the subsequent four moves in Figure 10. For this case we have

$$5 \cdot \frac{1}{4} + 6 \cdot \frac{1}{4} + 7 \cdot \frac{2}{4} \cdot \frac{1}{2} + 8 \cdot \frac{2}{4} \cdot \frac{1}{2} = 6.5$$

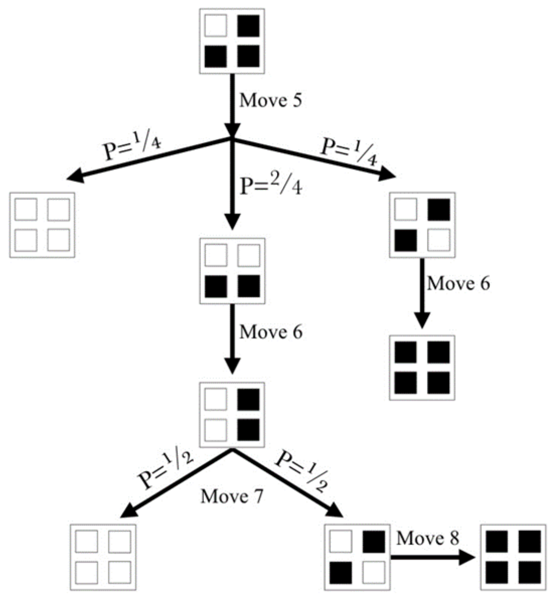


Figure 10: Probabilities for the (3-1) configuration

In order to calculate the mean value for this algorithm we multiply each probability with the number of the moves to obtain

$$\frac{2 \cdot 1 + 2 \cdot 2 + 4 \cdot 3.5 + 8 \cdot 6.5}{16} = 4.5$$

Having calculated the mean value for the algorithm as presented in Sections 3 and 4, we observed that the (3-1) configuration is solved in 5 to 8 moves. However, there are eight different (3-1) situations from a total of 16 situations, which means that the (3-1) situation seems to have a great impact on the mean value. Having that in mind, we devised a similar algorithm, which treats these configurations first. This “new” algorithm also consists of eight moves, which are: random – diagonal – consecutive – diagonal - random – diagonal – consecutive – diagonal. For this variation, the mean value for the worst-case scenario is 6. Hence, the mean value of the basic algorithm is less than that of the variation.

Bearing in mind the probabilities for each scenario, we construct a figure similar to Figure 10 with the only difference being in the enumeration of the moves.

For the case (3-1) we have

$$1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{2}{4} \cdot \frac{1}{2} + 4 \cdot \frac{2}{4} \cdot \frac{1}{2} = 2.5$$

Therefore, the mean value becomes

$$\frac{8 \cdot 2.5 + 8 \cdot 6.5}{16} = 4.5$$

The mean values for both algorithms are the same. This means that since this variation consists of eight moves as well, both algorithms are equivalent.

5.3 Rotating pentagon

The case of a rotating pentagon is slightly different from that of the rotating square. In this new challenge, the parameter of rotation actually affects the final solution. Here, the moves that can be made are three: Change of one random tile, change of two consecutive tiles and change of two nonconsecutive tiles. The situations of the pentagon are four: Monochrome, which means that all tiles have the same color (5-0), four same colored tiles and one different, two consecutive same colored tiles and three consecutive tiles with different color and finally two nonconsecutive same colored tiles and three different.

By examining the graph of the rotating pentagon, which we do not present because of its complexity, it became clear that the challenge of the rotating pentagon cannot be solved since every move potentially leads to all other (three) situations. On the contrary, the rotating square was solved because there was a situation (2-2 diagonal), for which a certain move led to monochrome while the same move performed a loop for the remaining two situations. The absence of such a property in the pentagon indicated that the rotating pentagon has nothing interesting to add to the challenge. It should be noted that the non-rotating pentagon can be solved by a series of 26 moves – a solution with no interest whatsoever.

6 CONCLUSION

Summing up, the black and white challenge was solved with an algorithm consisting of eight moves, a result reached by two different routes. This algorithm always completes the challenge, in a maximum of eight moves, regardless of the initial configuration. The features of the challenge are multiple and proved to be fascinating.

There were some queries that arose during our encounter with the challenge that were examined and answered. We proved that the parameter of rotation does not interfere with the solution; Actually, it makes the derivation of the solution an easier task. Furthermore, we examined whether it was possible to find another algorithm with a lower mean value than the one presented and, lastly, we tried to generalize the problem and study the case of a rotating pentagon.

INFINITY AND MATHEMATICAL PARADOXES

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ABSTRACT

"Don't just sit there counting stars, they are infinite, you will never finish" is what we were told when we were young. "I love you infinitely" is one of the first phrases one hears from his parents. What is truly infinite? In this paper, we discuss the multifaceted and complex meaning of infinity, which we approach through various mathematical paradoxes: An imaginary hotel with infinite guests, which varies from other hotels, envisioned by David Hilbert, a bag that fits an infinite number of ping-pong balls and a turtle that is so slow but at the same time unapproachable. All these paradoxes compose a journey to the most intricate but, at the same time, most peculiar mathematical concept: infinity.

INTRODUCTION

Since childhood, everyone gets in touch with the concept of infinity, without ever getting a clear explanation of what it actually represents. Many treat the universe, time, human consciousness or even their own lives as infinite, so that everyone can maintain their objective view. This blur that infinity brings with it, makes it not only unique, but also ambiguous and hard to define. The German mathematician David Hilbert elaborates on the concept that he engaged with throughout most of his life: "*The infinite has always stirred the emotions of mankind more deeply than any other question; the infinite has stimulated and fertilized reason as a few other ideas have".* (D.Hilbert, "On the infinite")
Its combination of simplicity and complexity was the reason that led humans to study it. Think, for instance, the natural numbers. If someone begins counting

them one after the other, he will soon realize that he will never reach a maximum number. What if he continued counting forever? Would he ever reach infinity? Even if he was able to count endlessly, he would find out that there is actually no maximum number. If someone believes he is able to find it, advise him just to add a unit to it.

Therefore, infinity is usually treated as something gigantic, yet this does not always represent the truth. Actually, it can also be identified as something minimal or really small. For instance, dividing a number by two incessantly, someone can construct an ever-smaller number. Based on this idea is the most recognizable Zeno's paradox, Achilles and the tortoise, founded. Will this division, however, keep on forever, or will a non-divisible number be reached?

The concept of infinity has been constantly bothering mathematicians, physicians, astronomers, philosophers, theologians, people educated or not educated since antiquity, while it is thought to constitute one of the most complicated issues human consciousness has to deal with. It has multiple applications in science, while it triggers existential and philosophical questions for life, universe or time. In this paper, mathematical paradoxes will serve as a means of explaining the history of infinity, a notion both ordinary and unique at the same time, as the iris in each eye [7].

Throughout this project, we try to give an explanation to basic elements that are related to the concept of infinity, such as the countable and uncountable infinite sets, their 1-1 and onto correspondence, Cantor's Diagonal Argument and the Continuum Hypothesis. "Hilbert's Hotel Paradox" is mentioned as an introduction to the countable infinite sets of numbers, as well as the comparison of sets developing an 1-1 and onto correspondence between them. The "Ping Pong Ball Conundrum" is also analyzed as a way to approach and deal with sequences that contain infinite elements. Furthermore, some of Zeno's most well-known paradoxes, such as "Achilles and the Tortoise" are discussed. In the last mentioned paradox, the use of sequences that tend to infinity is more obvious and clear and can be found in a larger quota. In this specific case, Zeno's flaw is detected, which coincidentally leads his initial reasoning to be dismissed.

INFINITY AND CANTOR'S ARGUMENT

In order to approach the inconceivable concept of infinity, answers to some fundamental questions are required. Initially, what does the concept of infinity represent? People's tendency to consider infinity as a huge number, slightly bigger than the biggest number they are capable of thinking, is reasonable, since everything observable in our world is finite and terminable. Even the number of atoms in the visible universe, despite it being huge, is finite. Only by realizing that it represents much more than a common number, can we delve into this intricate concept; a concept which just defines a number of elements. A never-ending number of elements, though.

Mathematicians decided to treat infinity not just as a simple meaning, but more as an entity, capable of getting added or subtracted remaining infinite or fall into categories. Simultaneously some infinities are infinitely bigger than others are [1, 2].

1. COUNTABLE INFINITE SETS

A first step in the attempt to get closer to infinity can be achieved by categorizing it. There are different levels of infinity. The lowest level of infinity is called countable infinity, to which the infinity of the natural numbers serves as a great example. Countable infinite sets are the sets, the elements of which can be put, the one after the other, in a sequence. This process actually indicates the numeration of the elements of an infinite set. Giving the first element the number one, the second the number two and so on, can be differently said as the correspondence of the elements with the natural numbers. Therefore, the fulfillment of the conditions for a correspondence to the set of the natural numbers stands also as a requirement for a set to be countable. Given these conditions, we are able to conclude that the set carries equivalent cardinality with that of the natural numbers. Cardinality of the set is called the number of elements in the set. Cantor named the countably infinite sets aleph-zero, $\aleph_0$.

2. UNCOUNTABLE INFINITE SETS

On the other hand, the set of all real numbers is an accurate example of this category of infinite sets. It is impossible to numerate even the real numbers between 0 and 1. Cantor’s diagonal argument will be conducive to this notion. The uncountable infinities cannot be put in order and are larger in cardinality than the countable ones. It is mind-blowing that the cardinality of all real numbers equals to that of the numbers in the interval 0 to 1 [3].

3. CANTOR’S DIAGONAL ARGUMENT

Suppose that we find a way to create a list of all real numbers from 0 to 1 and manage to correspond them with the natural numbers. If every natural number leads to only one different element of the set of the real numbers and there is no real number left unmatched, the set is countable.

1 ⟶ 0.1385445092…
2 ⟶ 0.9657432634…
3 ⟶ 0.4576537281…
4 ⟶ 0.7689372884…
5 ⟶ 0.8463526473…
6 ⟶ 0.2615345738…
Etc.

If a number that is not contained in the list can be found, then the set cannot be corresponded to the natural numbers and consequently is not countable. In his attempt to detect a not included number, Cantor followed efficiently the following path. He initially took the first digit after the decimal point from the first number of the list. Then he continued taking the second digit of the second number of the list after the decimal point, the third digit of the third number and so on. Thus, the number he obtained is 0.167954…
After realizing that it could not be proven that such a number was not contained in the list, he altered every of the infinitely many digits of the number by adding one, or by nullifying them if they were 9, forming the number 0.278065…
This number is undoubtedly not contained in the list, since it differs from any contained number in at least one digit. Consequently, this set is uncountably infinite. Cantor named this lever of infinity aleph-one, $\aleph_1$ [2].

4. CONTINUUM HYPOTHESIS

Georg Cantor wondered whether infinities other than $\aleph_0$ or $\aleph_1$ exist. He did not believe that, but he was unable to prove it. His hypothesis is still known as the continuum hypothesis. In the early beginnings of the 20th century, he regarded this assumption as the most significant unsolved problem in mathematics, yet not knowing its unexpected ending. A few decades later Kurt Gödel showed that it would be impossible to prove that the hypothesis was false, while in 1960 Paul J. Cohen showed that the formation of a valid proof of the hypothesis would not be possible to exist. Considering these two proofs, we are lead to the conclusion that unanswered questions are still looming in mathematics [4]. These facts of infinity are hardly conceivable, yet they only represent a small fraction of what the whole concept of infinity really bears.
The majority of elements that infinity brings with are opposed to common sense, yet they are all true. For instance, the amount of the odd numbers equals to that of the odd and the even numbers together. Initially someone will certainly believe that there are half odd than natural numbers; However this consensus is false. If two divides infinity, what remains is equally infinite. Furthermore, if a unit is added to it, nothing will really change. These elements are masterly revealed through Hilbert's Hotel paradox.

THE INFINITE HOTEL PARADOX

This Paradox talks about a hotel envisioned by the mathematician David Hilbert. Hilbert's hotel is full with an infinite amount of guests, each occupying one of the infinite amount of rooms. One day, someone enters the hotel and asks for a room. The receptionist rejects at first his request, thinking that the hotel is already full. Yet, a way could be found so that the new guest can stay at the hotel. The guest from room number one will move to room number two, the guest

from room number two will move to room three and so on. Every guest will move from room number n to room number $n+1$. As a result, the first room will remain vacant. Since the hotel has an infinite number of rooms and there is no "last" room, there will always be a next room to move to.
This operation can be repeated for every finite amount of people. If, for example, ten people arrive demanding a room, the receptionist would reposition the guest from room number one to room number eleven, the guest from room two to room twelve and the guest from room n to room number $n+10$ so that the first 10 rooms would become empty. Following the same pattern we could hypothetically find room for an infinite number of new guests. Unfortunately, the receptionist cannot instruct the person from room number one to go to room number one plus infinity, as infinity is not a number. So, a new way should be found for the infinite new guests to be placed in rooms [1].
If all current guests of the hotel move to rooms with even number tags, all rooms with odd number tags will remain empty. For this to be valid, each current guest of the hotel should move to a different room, which means that everyone that now has a natural number room tag should move to a unique odd number room. Additionally, all even numbered rooms should be occupied, which means that even and natural number sets must have the same amount of elements. The receptionist will move the client from room one to room two, the client from room two to room four, the client from room three to room six and so on. To generalize the process, the client in room n will go to room number $2n$. This results in all rooms with odd number tags becoming empty, leaving space for the infinite new guests.

Guest in room	goes to	Room
1	→	2
2	→	4
3	→	6
4	→	8
5	→	10
…	…	…
n	→	$2n$

New guest	goes to	Room
1	→	1
2	→	3
3	→	5
4	→	7
5	→	9
…		
n	→	$2n-1$

For two sets to have the same cardinality two conditions need necessarily to be fulfilled. The first condition is that one and only one element of the first set corresponds to one element of the second set. This means that there will not be two people in one room. This is called a 1-1 correspondence. The second condition is that no room should be left empty. Each element of the first set should correspond to at least one element of the second. If this is true, the correspondence is onto. So generally, a 1-1 and onto function between the two sets needs to be constructed. In this case, the two sets are the one of the Natural Numbers and the one of the Even Numbers. The first element of the natural number set is number one and it corresponds to the first element of the even number set which is two. Similarly two corresponds to four, three to six and so on, which shows that one element from the first set corresponds to one and only one from the second, and no number is left unmatched. This 1-1 and onto correspondence between the two sets leads us to the conclusion that they have the same cardinality.

Therefore, we found how to accommodate a finite and an infinite amount of passengers. Can we expand the problem even more? What will the receptionist do if an infinite amount of buses, each filled with an infinite amount of passengers arrived at the hotel? There should be a way to offer a room to all of them – that is by using prime numbers, as they are infinite [6]. So, in order to find infinite empty rooms for infinite buses of infinite passengers, the receptionist can move every current guest to a room of the first prime number, two, raised to the power of their current room number. For example, the person in room number five will be moved to room two raised to the power of five, namely room number 32. All passengers of the first bus will then be placed in rooms of the second prime, three, raised to the power of their seat number. This means that the person sitting on the fifth seat of the first bus will stay in the room number three to the fifth power, which is room number 243. Passengers of the second bus will be placed to rooms of the next prime, 5, raised to the power of their seat number. In the same way, powers of seven are used for the third bus, powers of eleven for the fourth one and so on. It is certain that two people will not end up in the same room. If two random prime numbers are taken, all their powers with natural number exponents differ. In other words, if a and b are prime numbers, $a \neq b$, and m, n natural numbers, then $a^m \neq b^n$ [7]. Note that with this technique, many rooms that are not powers of any prime number, such as 6 or 10, remain empty.

Room Movements	Passenger or Room Number						
	1	2	3	4	5	…	n
Hotel Guest	2^1	2^2	2^3	2^4	2^5	…	2^n
Bus 1	3^1	3^2	3^3	3^4	3^5	…	3^n
Bus 2	5^1	5^2	5^3	5^4	5^5	…	5^n
Bus 3	7^1	7^2	7^3	7^4	7^5	…	7^n
Bus 4	11^1	11^2	11^3	11^4	11^5	…	11^n
Bus 5	13^1	13^2	13^3	13^4	13^5	…	13^n
…							

Hilbert's hotel may be difficult to conceive, as it can never exist. Yet, it serves as a good introduction to infinity, in order to explore and try to understand the elusive and multidimensional concept of infinity that attracted, and still attracts, millions of people from all over the world. The countable infinity, which is illustrated in Hilbert's Hotel paradox, is also used in order to lead us to the majestically unexpected outcome of a paradox, using some common ping pong balls and a couple of bags with special properties.

THE PING-PONG BALL CONUNDRUM

Assume that a bag capable of holding an infinite amount of ping-pong balls exists. Suppose that we gather an infinite number of balls required to fill up this rather strange bag. Every ball is labeled based on the set of the natural numbers: 1, 2, 3, etc. and the experiment must be conducted in 60 seconds exactly.

It is about time we introduced George. George has the ability to complete an infinite amount of tasks in a finite time interval. During the first half of the time given, meaning during the 30 first seconds, George takes the first ten balls $(1, 2, 3, \dots, 10)$ and places them into the bag. He then moves on to take out of the bag the ping-pong ball with the number 10 on it. In the next 15 seconds, George gathers the ten next balls $(11, 12, 13, \dots, 20)$ and puts them in the bag, and then removes the ball with the number 20. During the next 7.5 seconds, he places inside the bag the balls $21, 22, \dots, 30$ and removes ball number 30. George keeps repeating this process always making sure that he adds ten balls while removing the 10th, and doing so in half the time than in the previous step. Therefore, the time intervals are being reduced every time by half:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

George has to repeat the same process an infinite amount of times in order to finish the process but this raises an intricate question. Will he ever finish? In order to answer this we have to take a closer look to the sum of the time intervals. By adding more terms this specific sum that contains the time periods approaches 1. Therefore, despite the fact that the process involves an infinite amount of tasks, it only takes the process one minute to be completed.
Another important question is the following: After the passage of 60 seconds, when the process is finally complete, how many ping-pong balls will one find inside the intriguingly spacious bag? Taking as a fact that each time George keeps adding 9 balls (places 10 inside and takes the 10th out) and that the process is repeated an infinite amount of times, there will be infinitely many ping-pong balls inside the bag. More specifically only the balls that were labelled with numbers that are a multiple of 10 will not be inside the bag.

1	2	3	4	5	6	7	8	9	~~10~~	11	12	13
14	15	16	17	18	19	~~20~~	21	22	23	24	25	26
27	28	29	~~30~~	...								

Suppose that during the time that George is placing balls inside the bag a friend of his, Anna, starts doing the same thing using another bag, but having a different strategy in mind. During the first 30 seconds, she takes the balls numbered $1, 2, 3, \ldots, 10$ and puts them inside the bag but instead of taking out the tenth ball as George did, she takes out the first (number 1). After 15 seconds, Anna places inside the balls that are numbered from 11 to 20 and removes the ball with the number 2. Moreover, after 7.5 more seconds have passed, she puts the balls 21-30 and discards the ball number 3. She continues every step by adding the next ten balls and removing from the bag the ball with the smallest number each time.
At first, both approaches may seem similar since both George and Anna add ten balls while removing one, but surprisingly the result is different. At the end of the 1 minute that they had in order to complete the task, they compare their bags. We do know that George's bag is filled with an infinite amount of balls but, incredibly, Anna's bag will have no ping-pong balls in it at all!
The whole task constitutes of infinite repetitions of the same process, adding and removing, so each number essentially corresponds to one repetition. This means that in Anna's' case the ball 33 was removed at the 33rd step, the ball 1243 was removed at the 1243rd step, the ball 157290 was removed at the 157290th step, and so on.

In what follows, we explain the reasons why Anna's bag is completely empty. Initially it is important to remark that the time intervals are reduced every time by half. Both Anna and George add the first 10 balls in the first 30 seconds (0-30s), then they add 10 more after 15 seconds (30-45s), in the next 7.5 seconds they add 10 more ping-pong balls (45-52,5s), etc. To proceed, we have to develop a 1-1 and onto correspondence between the set of the infinite number of ping-pong balls with the infinite set of the natural numbers, but since the balls we have are already labeled with natural numbers they are already placed in order, A=(1, 2, 3, ...) and N=(1, 2, 3, …).

To prove that Anna's bag will be empty in the end we essentially have to show that all of the balls are at some stage going to be extracted from the bag. A function that will give us the exact moment that each ball will be removed needs to be created. In order to do that we will have to once again form a 1-1 and onto correspondence between the set of the infinite balls A=(1, 2, 3, …) and the set of what contains and represents the time intervals B=(½, ¼, ⅛, …). The set of the time intervals can also be written as B = $(½, (½)^2, (½)^3, …)$ since the denominators of the fractions are powers of the number 2. The function is:

$f(x)=(1/2)^x$, where $x \in A$ which means: $x \in A \leftrightarrow (1/2)^x \in B$

This function proves that there is a moment that every ball Anna added in the bag will be removed from it. For example the ping-pong ball with then number 3470 will be taken out at the time interval which corresponds to $(½)^{3470}$ of the whole time of the experiment.

This indicates that after the passage of 1 minute, there will not be any balls left in Anna's bag. Were they to check and compare both bags on every step, what would happen? Any moment they checked before the 1 minute, they would both have the same number of balls, since they both add 10 and remove one ping-pong ball. At the end of the time when the infinite repetitions of the same task will have finished there will be a massive difference. It will be as if the balls in Anna's bag suddenly disappeared. This might be difficult for us to conceive since we are dealing with an infinite process performed in a finite time interval. It is physically impossible for us to check all the steps since, in fact, this task continues indefinitely.

In George's case, he keeps removing the 10^{th} ball, which leaves him with 9 balls left in the bag each time and since the process is repeated infinitely many times he has infinite balls in his bag. Whereas in Anna's case she keeps removing the smallest number every time (1, 2, 3, …) which indicates that sometime every ball will be removed, something that once again is based on the fact that the process continues infinitely.

An immensely important factor in this experiment is that we have in order and numbered balls based on the natural number set. Not only is it necessary for us to develop a 1-1 and onto correspondence between the set of the time intervals and the set of the ping-pong balls, but it is necessary that the balls are removed successively in Anna's case, beginning from ball 1. The existence of this type of organization and order is fundamental [5].

It was in 1953 when the mathematician John E. Littlewood described this rather mind-blowing paradox. Yet the relationship between humanity and infinity had already started centuries before, as it can be realized studying the famous paradoxes of the ancient Greek philosopher Zeno.

ZENO'S PARADOXES

Zeno was a philosopher in ancient Greece who tried to approach the concept of infinity, applying it to everyday phenomena. Thus, he constructed some statements contradictory to logic, his paradoxes. Zeno even managed to use the concept of infinity in his paradoxes to show that certain movements are impossible, the most famous of them being Achilles and the Tortoise.

In this paradox, there is a hypothetical race between Achilles and the Tortoise, but since the Tortoise is slower, it is given a head start. Suppose the tortoise's speed is 1 m/s and Achilles' speed is 10 m/s and the distance between the two is 100 m. According to Zeno, in order to reach the tortoise Achilles has to travel the initial distance between him and the tortoise, which is 100 m and it will take him 10 sec. During this time the tortoise will have moved another 10 m. Now Achilles has to travel these 10 m, but having done that, the tortoise will have moved another 1 meter. Zeno supports that since this process is repeated infinitely many times, the total distance Achilles has to travel is infinite, or in other words, it is impossible for him to finally reach the tortoise [9].

Although this conclusion is false considering common sense, can one prove it mathematically? We know there is an infinite amount of distances Achilles has to travel, but if we manage to prove that their sum is a finite number, we will have solved the problem. The sum of all terms is:

$$S_n = 100 + 10 + 1 + 0.1 + 0.01 + \cdots$$

This is called a series, each term of which is equal to the previous one multiplied by a specific number denoted by λ. The nth term of the sequence is denoted by α_n and equals the first term α_1 multiplied by r^{n-1}. The sum of the n first terms of a sequence is given by the formula:

$$S_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n = \alpha_1 + \alpha_1 * \lambda + \alpha_1 * \lambda^2 + \ldots + \alpha_1 * \lambda^{n-1}$$

By multiplying all terms of this series by λ we get:

$$\lambda * S_n = \alpha_1 * \lambda + \alpha_1 * \lambda^2 + \alpha_1 * \lambda^3 + \ldots + \alpha_1 * \lambda^n$$

Then, by subtracting S_n from both sides of the equation, we obtain

$$(\lambda - 1) * S_n = \alpha_1 * \lambda^n - \alpha_1 \Rightarrow$$
$$S_n = \frac{\alpha_1(\lambda^n - 1)}{\lambda - 1}$$

This is the general formula used to calculate the sum of the n first terms of such sequences. In this case, though, there is an infinite amount of terms, which means that n tends to infinity. We basically need to find the limit of S_n when n tends to infinity. In this case $\lambda = \frac{1}{10}$ but because $\frac{1}{10}$<1, the larger n gets the smaller the fraction becomes. So,

$$\lim_{n \to \infty} \lambda_n = 0$$

Now by completing the formula with what is known, the result is

$$\mathrm{S_n} = \frac{\alpha_1(\lambda^{\mathrm{n}} - 1)}{\lambda - 1} = \frac{100\ (0 - 1)}{\frac{1}{10} - 1} = \frac{1000}{9} = \mathbf{111.\bar{1}\ m}$$

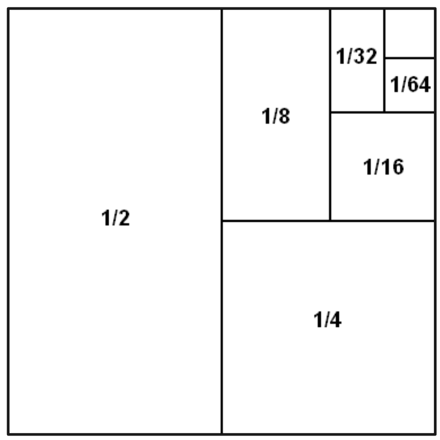
This is the total distance Achilles has to travel to finally reach the tortoise.
Similar to Achilles and the Tortoise is another paradox of Zeno, called the paradox of Dichotomy, which supports that movement is impossible. Suppose that someone is standing 1 meter away from a wall and his speed is 1 m/s. In order to reach that wall he first has to travel half of the distance, namely ½ meters, which will take him ½ seconds, he then has to cross half of the remaining distance, so another ¼ meters, which will take him ¼ seconds and so on. Zeno believed that because the man has to travel an infinite amount of distances, the time needed for him to complete the task would be also infinite. However, we know that it is possible for that man to travel 1 meter, but how can that be proved?
Since the distance is separated in parts, the total time needed for the man to cross it should be found. In other words, completing the movement will take

$S_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ seconds.

By using the same formula as in Achilles and the Tortoise, and because $\lambda = \frac{1}{2}$:

$$S_2 = \frac{\alpha_1(0-1)}{-0.5} = \frac{-0.5}{-0.5} = 1 \text{ sec}$$

Therefore, it is possible for the man to reach the wall and it will take him exactly 1 sec [8].
Another way of proving that the sum of S_2 is equal to 1 is by imagining a square. Firstly, half of the square is colored, then half of the remaining part and so on, meaning that we first paint ½ of the square, then ¼ of it, then ⅛ of it etc. As shown in the following picture, by repeating this process infinitely many times, we will finally get a fully colored square.



In conclusion, by calculating the sum of these series we proved that these notions of Zeno are wrong, since they are based on the false condition that the sum of an infinite amount of terms has to be equal to infinity.

EPILOGUE

Zeno's paradoxes depict the way people dealt with infinity in antiquity. Since then, the greatest minds of humanity struggled to pursue this mysterious, but attractive concept with the aim of proving some of these inconceivable facts we examined in this paper. One may wonder: Why do people try to engage with infinity, since the existence of Hilbert's Hotel, the strange bag with infinite ping-pong balls or the ability to complete an infinite amount of tasks in a finite time interval is impossible? The answer is simple. That is exactly what intrigues people and forces them to study it meticulously and try to answer the abundance of questions that it raises.
In this paper, we presented a variety of mathematical paradoxes with the aim of making the concept of infinity more approachable and easy for us to conceive. Although the paradoxes are initially prone to baffle us, since they counter rationality, the fact that they refer to everyday elements makes them graspable.

Thus, via understanding them, someone can be lead to useful conclusions on infinity. Firstly, the categorization of sets with infinite terms into countable and uncountable using Cantor's Argument is necessary, while the fact that all countable infinite sets have equal cardinality is successively shown in Hilbert's Hotel through 1-1 and onto correspondences. This paradox illustrates these correspondences, through an example of a hotel full with infinite clients that in some way resembles reality while maintaining a surrealistic tone. Common sense often goes against reality, as is shown in the Ping Pong Balls Conundrum. Finally, through Zeno's paradoxes a reference to infinity, mathematical series and sequences was presented, while the false conclusion to which Zeno was led was also mentioned.
At a first glance, infinity may seem as a humanly inconceivable concept, impossible to be limited or suppressed by the restrictions and boundaries of our minds. However, constructive time engaging with it, deliberation of practical mathematical problems and philosophical subjects that derive from the unfathomable being of infinity, while discovering the work of great mathematical minds that left their mark in the history of mathematics, may help someone approach and engage with this intricate, but mysterious concept.

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GRAPH COLORING

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ABSTRACT

This work focuses on the concept of Graph Coloring. We introduce the notion of the chromatic number and an algorithm that is used to calculate it in simple combinatorial problems that arise in Graph Theory (we present a scheduling problem). In addition, after mentioning the Induction Method and analyzing the proof of Euler's Characteristic and an extension of it, we present the five-color theorem proof. This theorem states that every planar graph can be colored by using at most five colors, so that no adjacent vertices share the same color. In conclusion, the notorious four-color theorem, together with the controversy over it, is mentioned and is illustrated by an attempt to color the map of Europe.

INTRODUCTION AND DEFINITIONS

Graphs are mathematical structures used to model pairwise relations between objects. They consist of vertices, which are connected with edges, while serving as a tool of imprinting and then studying information. This is exceptionally useful in many scientific fields associated with the binary code, such as Mathematics, programming telecommunications etc.

We first need to present some basic mathematical definitions in graph theory [1, 5], which are used later on in our project:

• A subgraph of a graph G is another graph formed from a subset of the vertices and edges of G. The vertex subset must include all endpoints of the edge subset, (but may also include additional vertices).
• The chromatic number of a graph is the least number of colors it takes to color its vertices so that adjacent vertices have different colors.
• A graph is planar if it can be drawn on a plane so that the edges intersect only at the vertices.
• A path is a sequence of consecutive edges in a graph.
• Two vertices are adjacent if they are connected by an edge.
• The degree of a vertex is the number of edge ends at that vertex.

In this paper, we present a proof for the five-color theorem, which states that any planar graph can be colored in such a way that no adjacent vertices have the same color, by using at most five colors. Obviously, the four-color theorem states that the same thing can be attained by the use of just four colors. Its proof is far more complicated and lengthy so we restrict to some comments in the final section.

A SCHEDULING PROBLEM

In a school, there are six different groups, which operate simultaneously after the end of school lessons. The groups are Mathematics, Physics, Astronomy, Theater, Ancient Greek and History. Figure 1 presents the student's names and the groups in which they participate.

Math	Physics	Astronomy	Theater	Ancient Greek	History
John	John	Annabeth	George	Rebecca	John
George	Rebecca	Mary	Bill	Mary	Skyler
Annabeth	Shae	Arthur	Shae	Bill	Mary
Skyler	Arthur			Skyler	

Figure 1: A Scheduling problem - Table

Knowing the abovementioned data, the school must find the minimum number of days needed so that no groups that share a student are scheduled to operate on the same day.

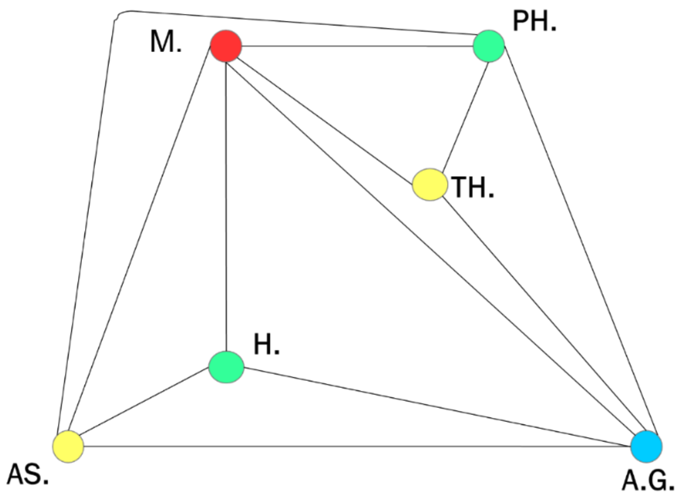


Figure 2: A Scheduling problem – Graph

In order to solve this problem, it is instructive to present the data with the aid of a graph. In this graph, as in Figure 2, the vertices represent the five groups while the edges that connect the vertices represent the groups that share students.

A first approach to solve a Graph coloring problem is the Welsh and Powell algorithm [2]. By using this algorithm, which we present directly below, one can find (most of the times, but not always) the chromatic number of a graph.

• Create a list with all vertices in descending order of chromatic number.

• Paint with the same color the first vertex of the list and all other vertices that do not connect with each other or with the first vertex. Delete the first vertex and all others that were painted with the same colors from the list.

• If the procedure is not over, repeat the second step.

By following this algorithm, one is indeed able to color the graph of the scheduling problem as is shown in Figure 2.

INDUCTION METHOD

Mathematical induction is a mathematical proof technique used to prove a given statement about any well-ordered set. Most commonly, it is used to establish statements for the set of all natural numbers.

It consists of the three following steps:

1. Show that the given statement holds for the first natural number of the set.
2. Make the induction hypothesis that the statement is true for the natural number $n-1$
3. By using the hypothesis, and making the correct moves, result in the fact that the statement is also true for the natural number n.

Mathematical induction can be informally illustrated by reference to the sequential effect of falling dominoes.

THE EULER'S CHARACHTERISTIC PROOF

Theorem: Let G be a simple, connected planar graph with V vertices, F faces and E edges. Then $V - E + F = 2$

Proof: The proof we follow is the one by induction [2, 4].
If G is acyclic, then $F = 1$ and the theorem holds because then G is a tree and $E = V - 1$. Otherwise, G has at least two faces and has a cycle. Let e be an edge that connects two different faces. Edge e is in a cycle. Deleting e from G and its planar drawing results in a graph G_0 with V vertices, $E - 1$ edges and $F - 1$ faces (since deleting an edge involved in a cycle merges the two faces on either side of it). By induction, we have that $V - (E - 1) + (F - 1) = 2$, and so, $V - E + F = 2$.

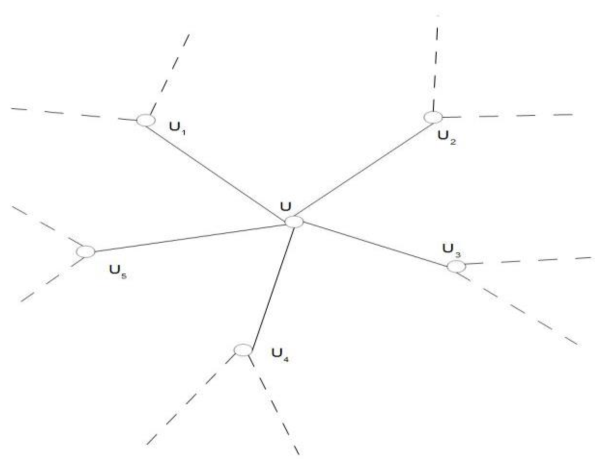
An extension of this theorem is the following corollary.
Corollary: If G is a connected planar graph and $V > 2$, then $E \leq 3V - 6$.
Proof: Every face is bounded by at least three edges. Counting the edges likewise, we find that the sum of all these edges is at least $3F$. Every edge bounds at most two faces. Therefore, $2E \geq 3F$ which gives: $F \leq 2E/3$. Because $V - E + F = 2$ we obtain $V - E + 2E/3 \geq 2$ which proves the corollary [5].

Obviously, the sum of the degrees of all vertices equals $2E$. Assuming that all vertices have degree greater or equal to six, we obtain $2E \geq 6V$ which contradicts to the corollary above. As a result, we conclude that in every planar graph there is at least one vertex with a degree of five or less.

THE FIVE COLOR THEOREM

Theorem: Any planar graph can be colored in such a way that no adjacent vertices share the same color, by using at most five colors.

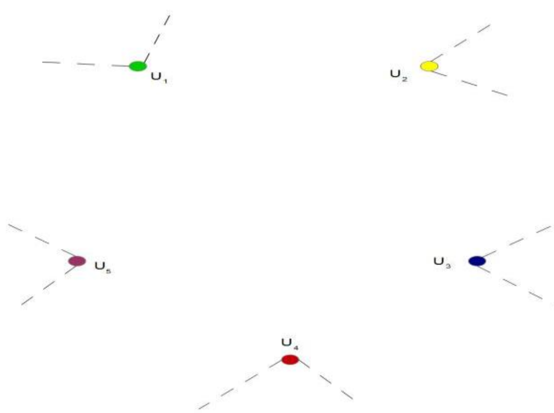
Proof: Let G be a random planar graph with n vertices. As we have already proved, in any planar graph there exists at least one vertex, which is adjacent to at most five vertices. In G we name this vertex u.



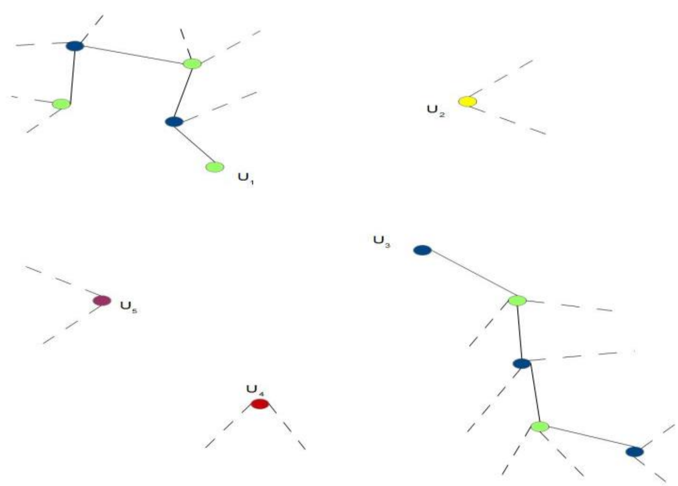
Then we remove u from G and assume by induction that there is a five coloring $for\ G - u$ ($n - 1$ vertices).

It is obvious that if u is adjacent to less than five vertices then we can find a five coloring for G, because u will be given any of the colors that have not been used to its adjacent vertices in $G - u$.

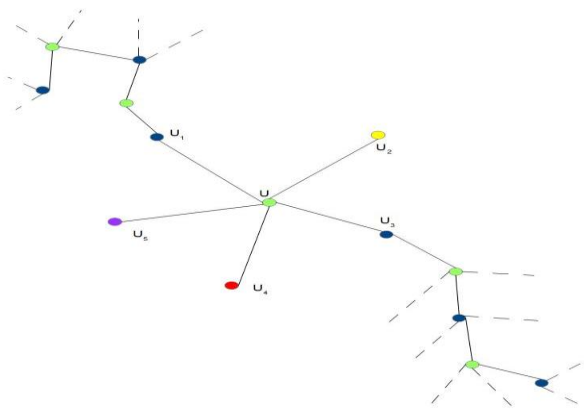
We denote the five adjacent vertices to u by the letters $u1$, $u2$, $u3$, $u4$ and $u5$ (which are adjacent with u in cyclic order) and color them with the colors $1,2,3,4,5$, respectively.



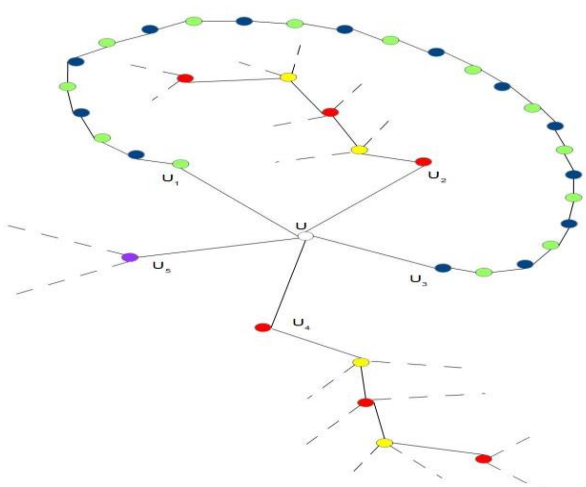
Now, we consider the subgraph $G1,3$ of $G-u$ consisting of the vertices that are colored only with colors 1 and 3 and edges connecting two of them.



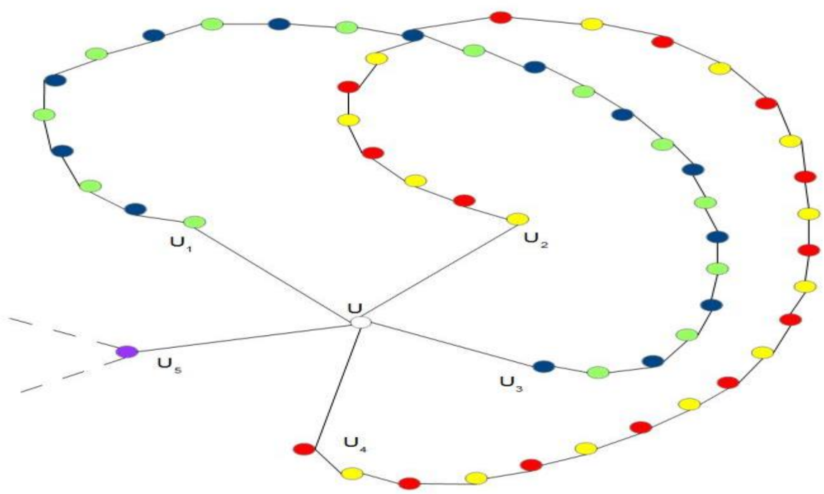
If $u1$ and $u3$ lie in different connected components of $G1,3$ we can reserve the coloration on the component containing $u1$, thus assigning color number 1 to u and finishing the task.



Otherwise, if $u1$ and $u3$ lie in the same connected component of $G1,3$ then we can find a path joining them, that is a sequence of edges and vertices painted only with colors 1 and 3. We will now take a look of $G2,4$ of $G - u$ consisting of the vertices that are colored only with colors 2 and 4 and edges connecting two of them and use the exact same method as before.



Then either we are able to reserve a coloration on a subgraph of $G2{,}4$ and paint u with color number 4, for example, or we can connect $u2$ and $u4$ with a path only containing vertices colored only with colors 2 or 4. The later possibility is absurd, as such a path would intersect the path we constructed in $G1{,}3$.



As a result there is always a 5-coloring for G**.**

THE FOUR COLOR THEOREM

The four-color theorem states that every planar graph can be colored with the use of four or less colors, so that no neighboring vertices are painted with the same color. The difference between this and the previously mentioned theorem may seem to be slight, but in fact is huge: many mathematicians have struggled during the last two-three centuries to find a solution-proof [3]. The first one came out in 1976 by Kenneth Appel and Wolfgang Haken, the second one in 1997 by Robertson, Sanders, Seymour, and Thomas (a simplified version of the previous proof based on the same idea) and the last one in 2005 by Georges Gonthier (check [6] and references therein). In all of these attempts, mathematicians did not manage to publish a strict proof of the theorem, but proofs in which the computer was one of the main tools: they restricted the possible cases with mathematics and then programmed the computer to check some thousands that were left. As a result, many members of the mathematical society do not fully accept these proofs. Because of the great length (over 50 pages) and complexity of the proof, we surely cannot present it here, but a configuration of it is shown in Figure 3. By creating a graph where the European countries are represented by vertices and neighboring countries are connected with an edge and by applying the Welsh and Powell algorithm we manage to find a four-coloring for the map of Europe.

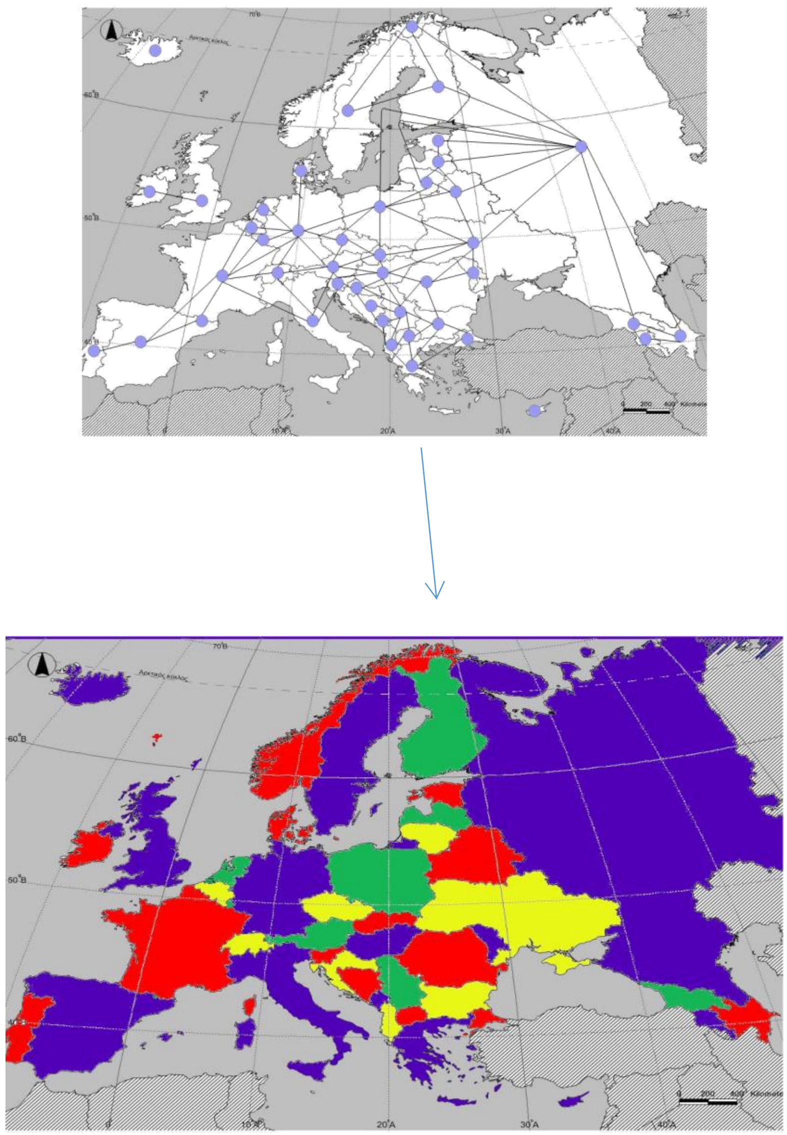


Figure 3: Four-coloring of Europe

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INSTANT INSANITY

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ABSTRACT

Instant insanity is a puzzle, a mathematical problem, which we present and solve. One is given four cubes, all sides of which are colored with different variations of the colors: red, blue, yellow and green. The goal is to place the cubes in such a way in order to form a tower, so that all four visible sides maintain all four colors exactly once. Is one capable of solving the puzzle without becoming instantly insane? Indeed, one may be too lucky and solve it on the spot. However, luck is not what we seek. Rather, a mathematical approach that guarantees a solution regardless of the initial configuration of the cubes.

HISTORY

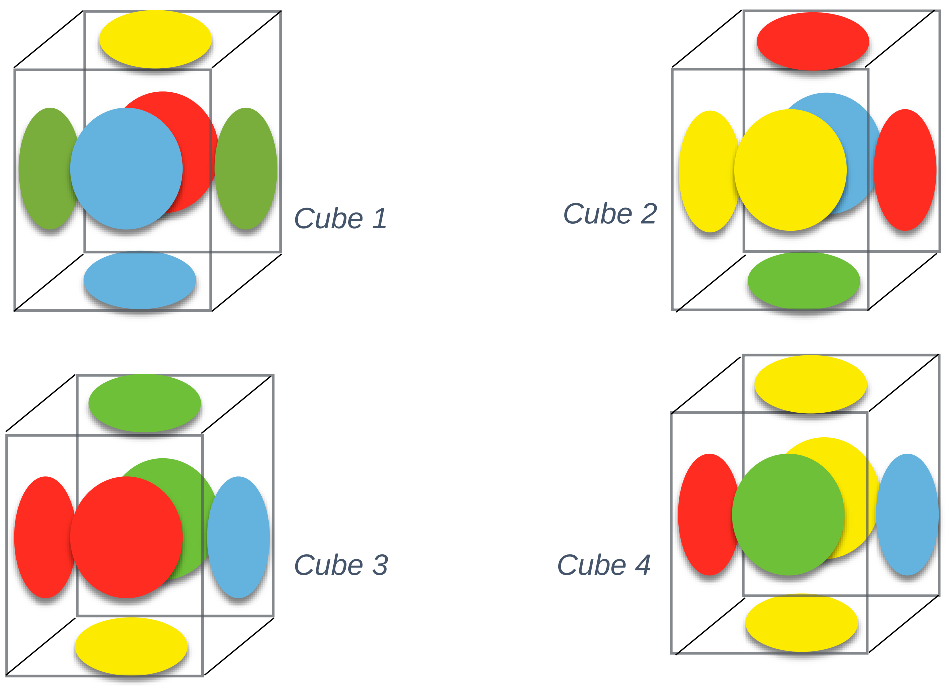
The “Instant Insanity” problem is a mathematic problem that was released in the late 1960s from the company Parker Brothers in puzzle form. However, there is a significant historical background behind this problem: It also went by the names “Katzenjammer”, “Groceries” and “The Great Tantalizer” [1, 5].

INTRODUCTION

The problem consists of four dice, each of them with six sides randomly coloured with different combinations of the colours: Red, Blue, Yellow and Green. Every die has all of those colours at least once. The task is to place the cubes in such a position in order to form a tower, all four visible sides of which will maintain all four colours. In this paper, we examine the initial configuration as depicted in Figure 1.

In the beginning, one tries to solve the problem through experimenting without any logic in mind. After a while, one realises that mathematics would actually speed up the process and make the solution luck-independent.

In the following Sections, we present our mathematical reasoning based on Graph Theory. We state our first thoughts on the problem, create graphs for the representation of each die, introduce the concept of a 2-factor of a graph and, finally, illustrate the way one identifies all such subgraphs in order to form the



solution to the problem.

Figure 1: Depiction of the dice

FIRST THOUGHTS

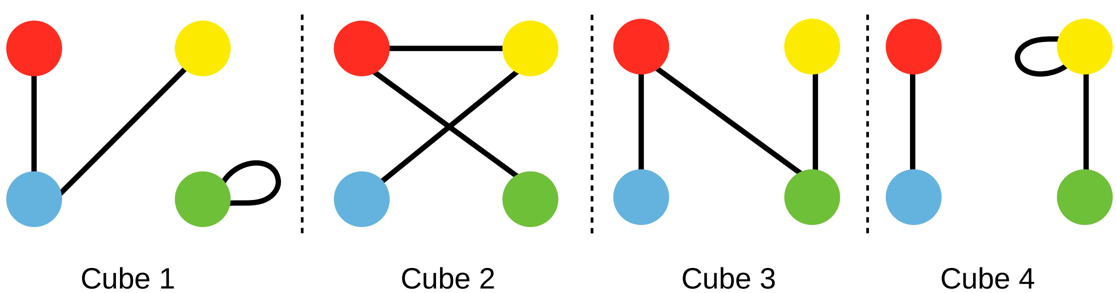
The depiction of the dice as presented in Figure 1 is visual but not at all handy for mathematical considerations. In order to study the problem carefully, we have to find a way to represent the dice in a more convenient way. To achieve this, one has to find the one characteristic of the dice that, whichever way we place them, does not change.

One may think that the fact that some colours are next to specific others is something that could lead to a solution. Consequently, we came up with the idea of utilising the vicinity of the colours of the dice. However, the vicinity of the colours of the dice is not helpful, as each colour has four others next to it, and that would make the orientation of the dice important. (Not to mention that the graphs would be extremely complicated.)

A far better idea is to utilise the opposite colours of each die. For example, as can be seen in Figure 1, colours yellow and blue are opposite for Cube 1. This proves to be the sole feature that remains unaltered whichever way we place the die. Our only concern in the final step of the solution will be the utilisation of the opposite colours in the hidden or the visible side of the tower.

GRAPHS

Our next step is to transform the information of the cubes depicted in Figure 1 into graphs. In these graphs, the vertices are four and they represent the four colours appearing in our dice. A line connecting two vertices (or a loop for one vertex), which is called an edge, shows that the two colours (vertices) are opposite one another [2]. The graphs depicting the opposite colours of each die



are illustrated in Figure 2.

Figure 2: Graphs for the four dice

We conclude that we have to combine these graphs in order to create a graph with all the information needed, altogether.

After having coloured the edges of the graphs of each cube with a different colour (Cube 1: Purple, Cube 2: Orange, Cube 3: Turquoise, Cube 4: Dark Red), we provide the final graph in Figure 3.

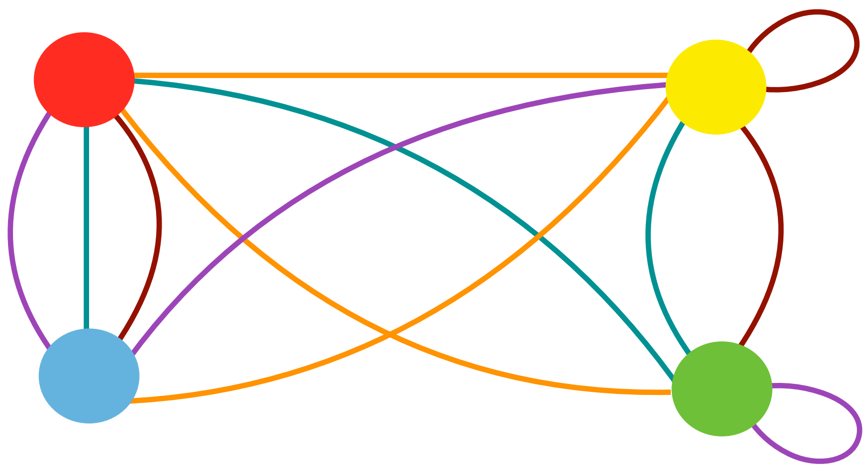


Figure 3: Graph combining all the information for all four dice

"2-FACTOR OF A GRAPH" SUBGRAPHS

The graph in Figure 3 represents the model of the problem. In order to solve the problem, we have to extract one subgraph out of the original graph, so we would solve one pair of opposite sides at a time. In addition, this procedure needs to be executed twice, as the final tower has two visible pairs of opposite sides. In this respect, we develop two points, which we call "rules", so that we can use our main graph in the right way. These rules are

1. Each vertex of the subgraph must be connected to two edges, as we are dealing with two opposite sides and our goal is for each side to maintain the four colours only once. It is obvious that we need each colour twice in total, one for each side. Thus, each vertex must have degree two.

2. We can use only *one* edge from each die at a time, because the tower we want to construct requires one pair of opposite colours from each cube. Therefore, the subgraph must contain exactly one edge from each cube.

Such a subgraph is a "2-factor of a graph" subgraph, the definition of which is the following [2, 3]:

An "x-factor of a graph" is defined to be a spanning subgraph of the graph with the degree of each of its vertices being x.

Note that a graph might have many different x-factors.

We can use the dice's information in the "2-factor of a graph" subgraph, only by replacing "x" with the number 2, meaning that each vertex has degree 2.

SOLUTION OF THE PROBLEM

As simple as it may sound, the identification of a 2-factor of a graph is a hard task. In what follows, we describe the way we can eliminate edges from our main graph, which we re-present in Figure 4, this time labelling every edge for quick reference.

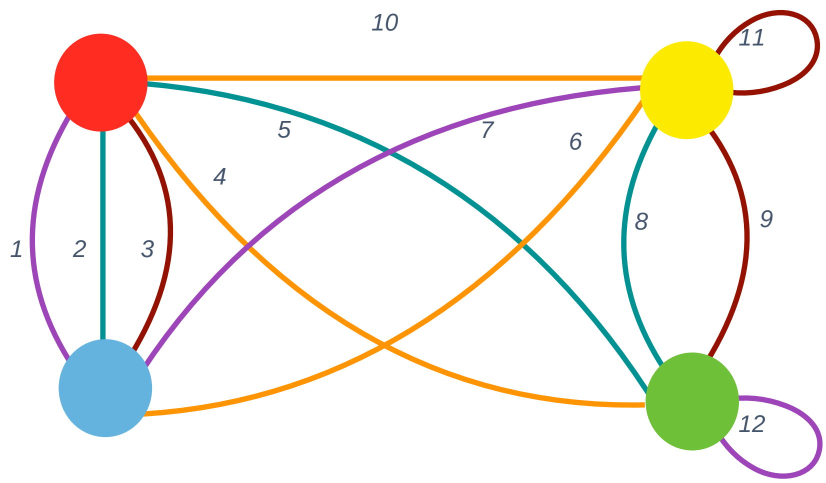


Figure 4: Main graph with labels for each edge

Based on the 2-factor of a graph theory, there must be two edges coming from each of the four basic colors. As a result, if we actually choose edge No.12 we end up eliminating edges 4, 5, 8, and 9 (because of the green color having degree 2) and edges 1 and 7 (because of cube 1). The remaining edges cannot result to a 2-factor. Hence, we eliminate the loop No.12, and with the same rationale, loop No.11 as well.

Having eliminated the two loops we proceed by choosing randomly another edge. We choose edge No.2 (blue). In this case, we cannot use edges 5 and 8 because of the second rule. Observe that we cannot use edges 1 and 3 neither because yellow and green vertices cannot have degree 2 without altering the degree of the already fixed red and blue. Therefore, we are left with edges 4, 6, 7 and 10 to increase the degree from 1 to 2 for both vertices red and blue. However, three of those edges come from the same cube and only one can be used. This means that edge No.7 must be included in the 2-factor subgraph together with either 4 or 10. Edge No.10 is excluded because that would mean that a dark red loop for the green vertex should exist. Hence, our only hope is edge No.4 that is realized since there exists edge No.9. These four edges constitute a 2-factor subgraph and as a matter of fact, it is the only 2-factor subgraph that contains edge No.2. Using the same methodology, we end up with the similar 2-factor subgraph as in Figure 5 (left) (the chosen edges are in bold) which is the only 2-factor subgraph that contains edge No.3.

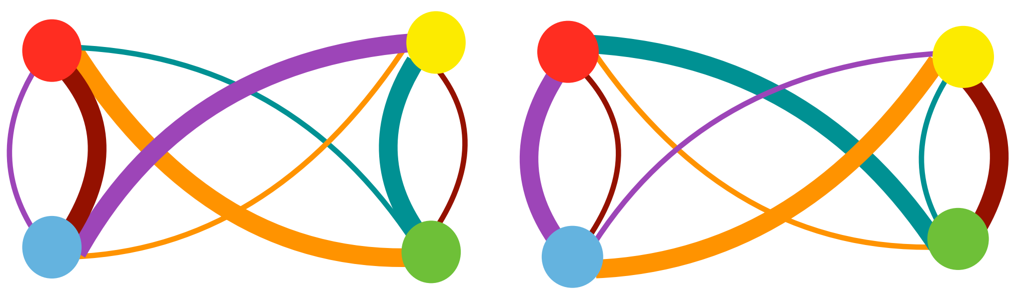


Figure 5: Both 2-factors of the main graph

Having chosen edges No.2 and 3, we now choose No.1. Obviously, edges 2 and 3 cannot be used. Edge 10 is eliminated first and we easily end up with the third 2-factor subgraph which we present in Figure 5 (right). Simple observations lead to the conclusion that there exists no other 2-factor subgraph.

It is important to note that such a thorough search for 2-factor subgraphs, as outlined above, is possible only in relatively small graphs. If the graphs contain more information or if one wants to guarantee that the search is indeed thorough, then an algorithm should be constructed.

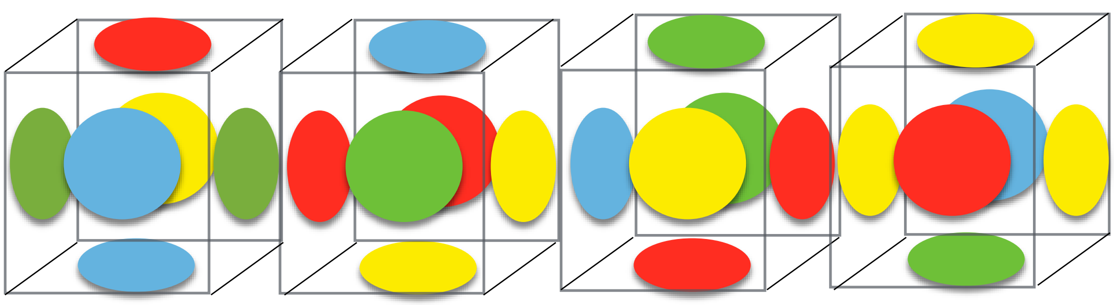
By utilising the two subgraphs as in Figure 5, we are able to construct the dice-tower correctly, with all four sides maintaining all four colours. It is impossible to use the 2-factor subgraph with edges 2, 4, 7 and 9 because it shares two common edges with the one in Figure 5 (left) and one common edge with the one in Figure 5 (right).

What we actually do is place the cubes the one above the other according to the first subgraph. After ensuring that these two sides maintain each colour once, we turn the dice, without altering their acquired position, and apply the information of the second subgraph making any necessary swaps. The problem is now solved and its solution is depicted in Figure 6.

CONCLUSION

In this paper, we have described the "Instant Insanity" problem while exploring how a solution can mathematically and systematically be found. We have also explained the need to use graphs and the way one solves the problem. Of course, the solution, as in Figure 6, is the solution for the specific configuration of the cubes. In any other situation, in which even one side is differently coloured, the solution will be completely unalike.

Figure 6: The solution to the Instant Insanity problem



The Instant Insanity problem constitutes a typical example of a graph theory application [2, 3, 4], while being an excellent opportunity for someone to exercise his mind by discovering the mathematical way of succeeding a task. An efficient algorithm for the general case of n cubes is presented in [6] and a combinatorial analysis of some special cases and generalisations of the problem are examined in [7].

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SHORTEST PATH PROBLEM

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ABSTRACT

Say it is a typical weekday and you are on your way to work or school. You probably follow a certain route you have been used to. It usually gets you there on time, but often there are traffic jams and sometimes you are in a hurry and end up arriving late. You begin to wonder: what is the fastest way to get to my destination? How can I find the shortest route?
This is exactly the problem we seek to resolve in this paper, where we examine algorithms that find the shortest path between predetermined locations. Through concrete examples and practical simplifications, we aim to explain and analyze the most well-known algorithms of said type. Moreover, we report on their applications in our everyday lives in various sectors including navigation, telecommunications and data transfer. Thereby, we demonstrate their pervasive use in the modern world and the great value they hold across all scientific fields.

INTRODUCTION

In this paper, we examine a well-known problem derived from Graph Theory called “Shortest Path Problem”. The problem is the following:

A traveller finds himself in a city and wants to visit another town. In order to reach his destination he has quite a few options as far as paths are concerned, many of which pass through other cities. However, these different routes turn out to have differences in terms of fuel costs, tolling and distance. The traveller wants to spend the least amount of fuel/money possible. How is he going to figure out the optimal path to follow to get to his destination?

Of course, there are many variations and different ways of expressing the problem. The strictly mathematical formulation is the following:

The shortest path problem refers to the determination of the path between a source vertex (starting point) and a destination vertex with the smallest total edge weight.

This problem has numerous modern applications in fields such as public transport, telecommunications, computer science and geographic navigation systems.

BASIC GRAPH TERMINOLOGY

Before moving to the presentation of the problem's solution, we quickly make a brief mention of some of the fundamental terms of Graph Theory.

Graph: A graph, see Figure 1, is a diagram of points and lines connected to the points. It has at least one line joining a set of two vertices with no vertex connecting itself. A graph 'G' is defined as G = (V, E), where V is the set of all vertices and E is the set of all edges in the graph.

Vertex: A vertex (or, node) is a point where multiple lines meet. In Figure1, a vertex is denoted with the letter 'A'.

Edge: An edge is the mathematical term for a line that connects two vertices. Multiple edges may begin from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge. In Figure 1, A and B are two vertices and the link between them is called an edge, denoted AB.

INITIAL SOLUTION ATTEMPT

We wish to determine the shortest path between the source vertex, A, and the destination vertex, H. At first, we decided to try choosing the edge with the smallest weight connected to the node at which we were at the time, beginning from the starting node. We thought that by selecting each time the path from the vertex that was our current to its closest neighbor, we would be following the shortest route every step along the way. Moreover, in our methodology, we set a basic rule; it would be forbidden to return to a node we had already visited.

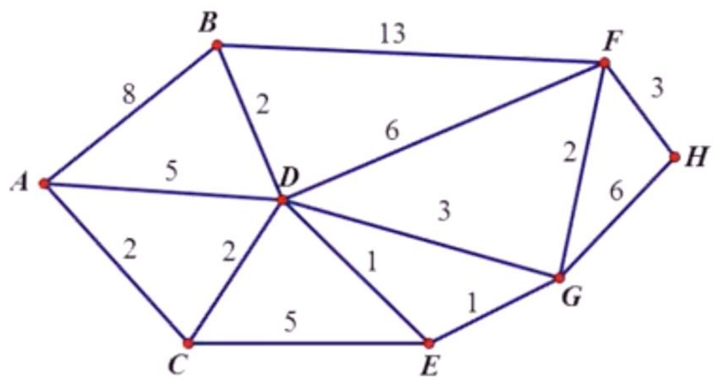

Figure 1: An example of a graph

In Figure 1, we start from node A and select among the edges to the vertices B, C and D the one with the smallest weight (as these are the vertices directly connected to A). In this case, we choose the edge AC and proceed accordingly.

Although our initial thought process has some valid points, it is not correct. As it turns out, the paths that might seem optimal at some point during the examination of the graph do not necessarily make up the optimal overall route from the source node to the destination. We later discovered that this technique is called a “greedy” strategy and while it fails to find the shortest path, it is employed in the solution of other graph related problems.

To prove that this method does not lead to determining the shortest path, we take the nodes B, D and F of the previous system and examine them as a separate isolated graph. Suppose we start from D and we want to reach F. According to our method, we would go from D to node B as the edge joining them has a weight of 2, as opposed to that which connects D to F that has a weight of 6. Thus from B we would then go to F. The route followed, however, would have a total edge weight of 15. If we had just moved from D to F directly, disregarding the method we had invented which suggests always choosing the optimal route at every moment without thinking further ahead, the path would have been shorter. In conclusion, the main flaw with this attempt was that upon the selection of a certain path there could be no revisiting alternative routes.

DIJKSTRA'S ALGORITHM

A quick and efficient solution to the shortest path problem is the use of Dijkstra's Algorithm, an algorithm conceived by Edsger Dijkstra [4, 5] in 1956. This is an algorithm that determines the shortest path from a common starting point in a (directed or not) graph with non-negative weights on the edges (single-source shortest path problem). Dijkstra's algorithm is greedy, meaning that at each step it selects the locally optimal solution, until finally composing the optimal solution [2]. The procedure of the algorithm is described by the pseudocode in Figure 2 [3].

In this pseudocode, we aim to find the shortest path between the source vertex and vertex **v**. In order to understand the process of the algorithm' function we break it down to instructions, explaining every step (the terms "node" and "vertex" are used interchangeably).

Firstly, we create a vertex set, named **Q**, containing all the currently unvisited nodes/vertices; the ones the algorithm has not examined yet.

To each vertex **v**, we attach a distance label **dist [v]** with a value of zero for the source node and infinite value to all the rest. We also place a previous label to every node **v** (**prev [v]**). This has undefined value for all the nodes. This label is needed for calculating the desired path in the end of the process.

While set **Q** is not empty, meaning there are unvisited nodes to examine, we select the vertex with the smallest distance from the source, say vertex **u**. Since we examine this vertex, we remove it from the unvisited set **Q**.

For every vertex **v** neighbouring of **u** we create a new variable called **alt.** This is defined as the sum of the distance of **u** plus the length of the edge that connects **u** with neighbour **v** {alt← dist[u] +length (u, v)}. This variable essentially represents the distance between the source vertex and vertex **v**.

```
function Dijkstra(Graph, source):

   create vertex set Q

   for each vertex v in Graph:            // Initialization
      dist[v] ← INFINITY                  // Unknown distance from source to v
      prev[v] ← UNDEFINED                 // Previous node in optimal path from source
      add v to Q                          // All nodes initially in Q (unvisited nodes)

   dist[source] ← 0                       // Distance from source to source

   while Q is not empty:
      u ← vertex in Q with min dist[u]    // Node with the least distance will be selected first
      remove u from Q

      for each neighbor v of u:           // where v is still in Q.
         alt ← dist[u] + length(u, v)
         if alt < dist[v]:                // A shorter path to v has been found
            dist[v] ← alt
            prev[v] ← u

   return dist[], prev[]
```

Figure 2: Pseudocode of Dijkstra's algorithm

For every neighbour **v** of node **u,** we compare the calculated **alt** with the distance assigned to it by the distance label **dist[v].** If **alt** is smaller than **dist[v],** we proceed to change the distance label assigning it a value equal to **alt**. A shorter path to node **v** has been found. The previous node label of vertex v **prev[v]** becomes **u**, exactly because the shortest path we have so far found to vertex **v** comes from vertex **u**, making **u** its previous vertex.

We explain the procedure in simpler terms: Essentially, the algorithm relies on three variables, also known as labels: the **distance** variable **dist[v]**, the **alternative** variable **alt[v]** and the **previous** variable **prev[v]**. In a given graph, the source vertex is already defined and indicated before the implementation of the algorithm. The vertices that are directly connected to the source with just one edge are known as neighbouring nodes of the source or simply **neighbours**.

At the very beginning of the algorithm's process, two separate sets of vertices are created: the visited and unvisited set. The former contains the vertices that have been examined and the latter the ones that have yet to undergo inspection by the algorithm's process. Then, the distance label is assigned to all the vertices in the graph according to the following plan: For the neighbours of the source the distance variable assumes a value equal to the weight of the edges that link them to the source vertex, respectively. For the rest of the vertices in the graph that are not adjacent to the source, the distance label is set as infinite at this stage.

After having set the dist[v] for all the nodes of the graph, the algorithm's operation continues with the selection of one of the neighbours of the source to (we assume that this is denoted by the letter **j**). The neighbour chosen is the one whose distance label has the lowest value. It is important to note that this choice does not necessitate that the other neighbours of the source will not be examined later on. On the contrary, they will, unless the shortest path has already been determined. The vertex that has been selected is removed from the "unvisited" set and added to the "visited", as will happen with every vertex examined subsequently.

Now, the algorithm looks into the nodes that are connected with vertex **j**. We denote every one of those as **t** (we select a random one but it actually represents them all). At this point, the "alternative" label comes into play. This is defined as the calculated accumulative sum of the weights on the edges that comprise a certain path from the source vertex to a given vertex in the graph. Thereby, for vertex **t** the alternative label is given by the addition of the weight of the edge that connects vertex **j** to the source, meaning the "distance" variable of vertex **j**, plus the weight of the edge that interconnects **t** with **j**. In essence, to calculate the **alt** of any vertex all we have to do is add up the weights of all the edges that construct the path we have followed from the source to this particular vertex.

Moving on, the algorithm compares the distance label of vertex **t**, assigned at the beginning, with the newly calculated alternative. If the alternative is smaller than distance label, then the latter is changed and assumes a value equal to alt. Virtually, if alt < dist, a shorter path to the specific vertex has been found and that is why the distance label is replaced. As mentioned, the calculation of the "alternative" label depends on the path that has been followed to reach a certain vertex beginning from the source. Since there can be multiple different routes leading to a vertex starting from the source, during the algorithm's process the alternative variable of the vertex assumes numerous different values as these paths are being tested. The comparison with the most recently assigned distance label is thus repeated until all the options have been explored and the shortest path to this vertex has been found.

Following this process, the distance label of the vertex will undergo numerous improvements until assuming the smallest possible value. Applying these steps to all the vertices, the algorithm determines the shortest path from the source to all the vertices in the graph including the destination vertex. The “previous” label is used during the entire procedure to keep track of the path that is being followed throughout the graph. Thereby, the derivation of the path is always reported every step along the way. For instance, in our aforementioned example, the previous label for vertex **t** is **j** since we arrived at vertex **t** via vertex **j**.

RUNNING TIME

Dijkstra’s algorithm is one of the most efficient algorithms applied to solve the Shortest Path Problem. Its running time is relatively low when compared to the running time of other algorithms, especially in large graphs. Bounds of the running time of Dijkstra’s algorithm on a graph with edges **E** and vertices **V** can be expressed as a function of the number of edges, denoted **|E|**, and the number of vertices, denoted **|V|.** The size and complexity of the graph are the deciding factors and they are represented mathematically by certain variables, which are used to indicate these properties of a graph, and are included in the function that calculates the algorithm’ running time.The running time also varies based on the way the vertex set **Q** is implemented (visited set). In other words, the necessary time is different depending on whether a priority queue is used and how that takes place.

Evidently, the algorithm’ running time increases greatly as the number of edges and/or vertices in the graph grows. However, Dijkstra’s algorithm remains one the fastest algorithms that can be used to determine a shortest path, adjusting the relative speed to performance strength [1, 3].

RESTRICTIONS IN USE

As mentioned in the introduction, Dijkstra’s algorithm cannot be applied in graphs where the weight on any of the edges is a negative number. The explanation for the method’s failure in such cases is as follows:
Recall that in Dijkstra’s algorithm, once a vertex is marked as “visited” (and out of the “unvisited” set), the algorithm has found the shortest path to it, and will never examine this vertex again, assuming that the path developed to it is the shortest. This assumption allows the algorithm to reject certain path options and reduce the field of different routes that must undergo examination. However, with

negative weights this might not be true. Consequently, that assumption cannot be safely made. As a result, the algorithm would need to perform the examination of all the possible paths so as to avoid inaccuracies, practically losing its selective nature and being rendered to a brute force algorithm that tries out each and every option. In other words, Dijkstra's algorithm relies on one "simple" fact: if all weights are non-negative, adding an edge can never make a path shorter [1, 2, 3].

EXTENSIONS

Research into the Shortest Path Problem, as well as Dijkstra's algorithm and their various implementations, is still being conducted and numerous improvements and extensions are being investigated and tried out. For instance, a new probabilistic extension of Dijkstra's algorithm is presented in [6]. Its developers claim that it can be applied to accurately simulate realistic traffic flow and driver behaviour and aid in the improvement of urban circulation. This notable extension introduces probabilistic changes in the weight of the edges as well as the decisions when choosing the shortest path.

On the other hand, recent developments in the area of route planning in transportation networks have led to algorithms and methods that are up to one million times faster than Dijkstra's algorithm. Ideas, algorithms, implementations, and experimental methods can be found in [7] and in references therein.

MAP EXAMPLE

The weight on the edges can represent different variables in real life implementations of the problem, including time, distance, monetary cost, fuel consumption, gas emissions etc. Depending on what the edge weights stand for, the algorithm leads to the determination of a different shortest path in the same given graph. We examine an actual example to illustrate all the above.

In Figure 3, we present a civil road map of Bucharest containing the depiction of a number of routes between two designated locations: the RIN Grand Hotel, where the Euromath Conference was held this year, and the Henri Coandă International Airport, Romania's busiest airport. Remarkably, the route that is shorter in terms of distance, measured in kilometres, is also the most time-consuming. As a result, the shortest path does not constitute the ideal choice because we are taking into account the two coinciding variables of time and distance simultaneously. The quickest route, taking 44 minutes, is actually the second longest.

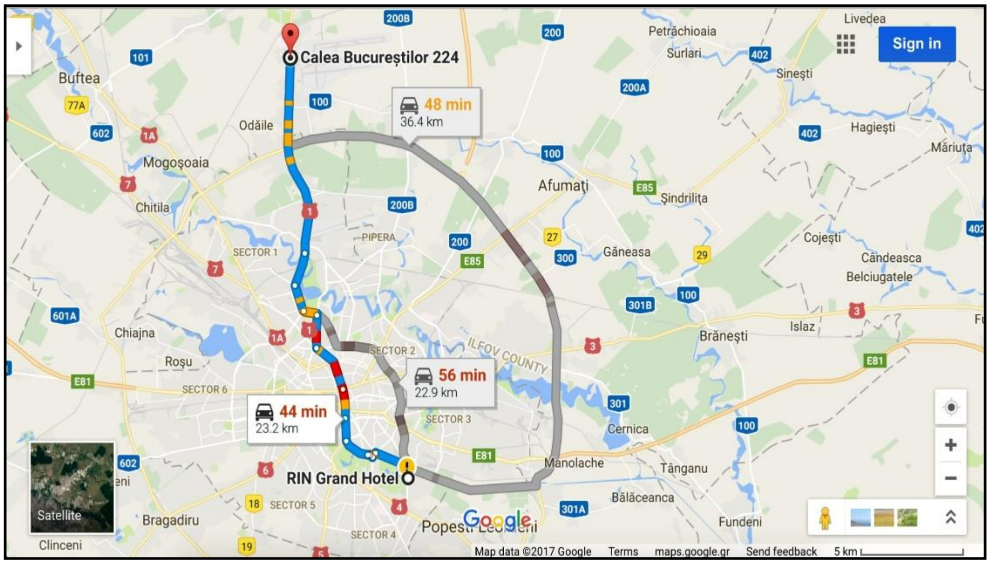

Figure 3: Bucharest map

With the appropriate transformation, the network of alternative routes/paths reflected above, can be converted to two separate graphs as in Figure 4: one where the edge weight refers to time (left one) and another where it represents distance (right one). Although the graph is essentially the same, the difference in the edge weights due to their symbolising different information, results in a new shortest path. In the first graph, the shortest is S_1 = {A, B, D, E, F} with a total weight of 44 whereas in the second one the shortest path turns out to be S_2 = {A, B, D, F} with the accumulated weight of 22.9.

In conclusion, this indicative example demonstrates that the shortest path on any given graph differs based on the criteria we select. Often, we can use a wide range of alternative criteria or combine them thanks to the advanced possibilities modern technology offers us. In navigation devices, for instance, we can adjust the parameters that are taken into consideration when determining the recommended routes between the location we set as starting point and the destination, to avoid main streets, where we are likely to encounter traffic, or exclude routes that require tolls. In addition, the GPS device automatically processes the data it receives from its connected satellites regarding the state of circulation and reroutes the suggested course in case of accidents or road works, proposing alternative paths and shortcuts.

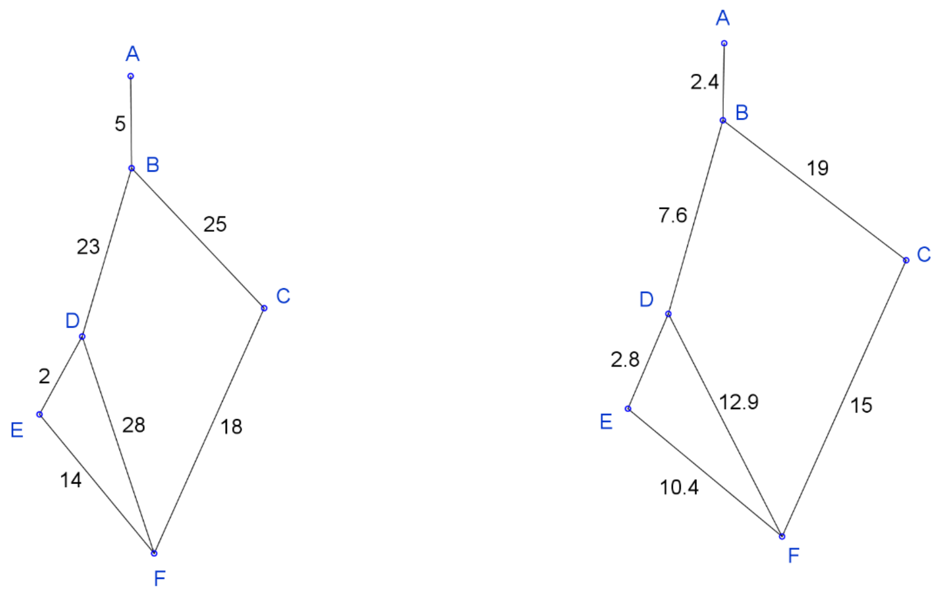

Figure 4: Two graphs stemming from the Bucharest map

Note: These graphs are not entirely precise representations of the routes depicted on the map but relatively vague illustrations for demonstrative purposes. The individual edge weights have not been attributed with accuracy although care has been shown so as for the total accumulated edge weights of the three different possible paths to be exactly those of the real life scenario, in terms of time (in minutes) and distance (in kilometers) respectively. Locations corresponding to vertices:

A➲ Henri Coandă International Airport

B➲ Intersection between Calea Bucureştilor and DNCB Șoseaua Odăii

C➲ Cemetery Cimitirul Pantelimon 2

D➲ Piața Charles de Gaulle, Aviatorilor

E➲ Piața Unirii

F➲ RIN Grand Hotel

A very helpful interactive visualization of the way the algorithm works can be traced in [8]. To watch the process, select a vertex to be your source and type its number in the cell titled “Start Vertex”. Subsequently, click “Run Dijkstra”. The animation speed can be adjusted by using the corresponding bar at the bottom of the page.

ALTERNATIVE SOLUTIONS TO THE SHORTEST PATH PROBLEM

Dijkstra’s solution to the shortest path problem seems to be the most used and popular solution. However, there are other solutions, which may be more complicated or more specialised that can be more helpful or faster compared to Dijkstra’s. For instance, Bellman-Ford’s algorithm aims to solve problems where the edge weights can be negative. The Floyd-Warshall algorithm can be described as a fast and better choice for comparing all the pairs of vertices in graphs with many edges. In contrast, Dijkstra’s algorithm is optimal in a graph with a relatively smaller number of edges, which are not negative [1, 2, 3].

Yet another way of finding the shortest possible path is the brute force approach. This solution to the problem, as the name implies, virtually calculates every single path and then compares the total sum of the weights in order to pick the optimal route. A clear advantage to the application of this approach is that it is very informative: not only does one find the shortest path when he employs the algorithm, but he may also find the 2nd, 3rd or even the worst of all paths. In addition, there are no restrictions or limitations to its use. Nevertheless, the brute force method lacks the selective nature of Dijkstra’s technique (meaning the step where Djikstra’s algorithm compares the assigned distance value with the calculated distance and replaces the initial label if necessary). This deems the method cumbersome and time-consuming. Moreover, the number of possible routes on a graph increases exponentially with each vertex; as a graph expands, employing this specific solution becomes a daunting task [12]. In fact, unless a graph is extremely simple and succinct, it is nearly impossible for any human being to compute the final solution. Indeed, there are much faster ways of finding solutions.

COMPUTER SCIENCE (CS) ALGORITHM OUTPUT

We have to keep in mind just how large the graphs on which we apply these algorithms are. In this paper, we have used and constructed very simple graphs, meaning they consist of few vertices. Employing any type of algorithm on an accurate graph of a city's road network by hand would be tedious, intricate and far too complex to demonstrate.

Therefore, it comes down to Computer Science as a discipline to constantly improve these algorithms and to bridge the gap between our capabilities and those of computers. In order for any algorithm to be applicable in digital form, it has to be written in code (the language of computers), which looks surprisingly a lot like the mathematical algorithm of Figure 2. This can be accomplished through a lot of different software systems, like java and python. Moreover, a map is required with all the paths and necessary information. The output would look like something as in [11]. This specific example uses the brute force technique to find the shortest path from Washington DC to Los Angeles, via the railroad network in the USA. This example is accurate and helpful since it shows how the algorithm gradually starts to compare all vertices on the graph. When a computer or a GPS does this in the real word it is done far more faster, the example above it is far slower purely for demonstrative reasons.

REAL WORLD APPLICATIONS

Shortest path algorithms are implemented in many sectors beyond the mathematical world. When you want to find your way in the city and decide to use Google maps, you actually employ a program that determines the shortest path between your location and your destination. These algorithms are also essential to the function of telecommunication networks and data transfer. Specifically, when watching a video, or loading a site, the data uses the shortest possible path from the server to your device (your IP Address). This is where path-finding algorithms come into play: by choosing the optimal satellite(s)/cable line, there is minimum waiting time once you click on a link/make a call. Given how often these incidents happen, it is safe to say that path finding algorithms have, in a way, shaped our society and facilitated everyday matters, such as driving and surfing the web. One might even say that modern societies, which are characterized by an abundance of information, cannot function properly without them.

Although GPS devices and telecommunications are the most prevalent way of how we utilize these algorithms, there is a plethora of other applications. For example, in electrical circuits and urban planning, Djikstra's algorithm can be used to simulate the flow of electricity and traffic accordingly. This way, once we have created the circuit it will prevent the system from going haywire. In the latter case, such uses are vital since the algorithms can help find ways of building up-and-coming neighbourhoods or cities, which are looking to expand outward in a controlled and efficient manner [10].

CONCLUSION

In conclusion, the Shortest Path problem is one of the many important as well as interesting problems of Graph Theory. In this paper, we focused primarily on Dijkstra's algorithm, which is the most widely used method but we also took care to include some other techniques of finding the shortest path. In the process, we realised the incredible value of the solutions provided for this problem and their vast spectrum of practical applications in the real and digital world as well as other scientific fields such as Information Technology, Computer Science and Physics. No matter whether the weight on the edges of the graph represents monetary cost, distance, time, length of intercontinental communication wires or even planetary orbits, the method of finding the optimal route is always applicable.

From the planning of flight itineraries and train routes, the function of portable navigation devices that cater to the need for fast commuting of millions of people worldwide and the instant transfer of all kinds of data from the globe's largest server computer to each individual laptop at the click of a mouse, algorithms for determining the shortest path like Dijkstra's have contributed to numerous essentials advancements in science and technology and have enabled humans to greatly improve their everyday life and expand their capabilities and horizons thereby shaping the modern world.

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FROM LO SHU THROUGH SUDOKU TO KEN KEN

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ABSTRACT

In my talk I will speak about magic squares. Apparently the first appearance of a 3 by 3 magic square is connected to an ancient Chinese legend about the Lo Shu turtle which emerged from the Yellow River with a very peculiar pattern on its shell.

I will present this legend briefly. Then I will pass to the Renaissance and speak about a magic square invented by a German artist and scientist Albrecht Duerer. Finally, I will show how magic squares come up in puzzles. Actually, this was the main motivation for my research. I like master mind like challenges and puzzles and I want to share my enthusiasm for them in my presentation.

My presentation is about mathematics magic squares which is a part of my math project about the jigsaws. I really enjoy this part of math so that's how it all started. I want to show and explain the action of the magic squares.

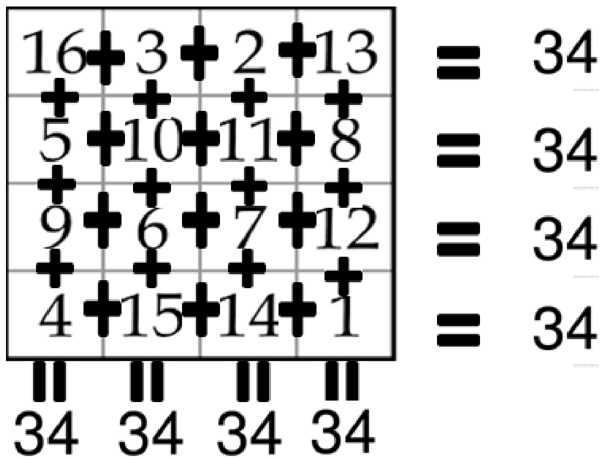
The first aspect is a Magic Square. This is a chart with the same numbers of rows and columns completed with nonrepeted digits that in every row, every column and across the sum is the same.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Example of a magic square dimension 4.

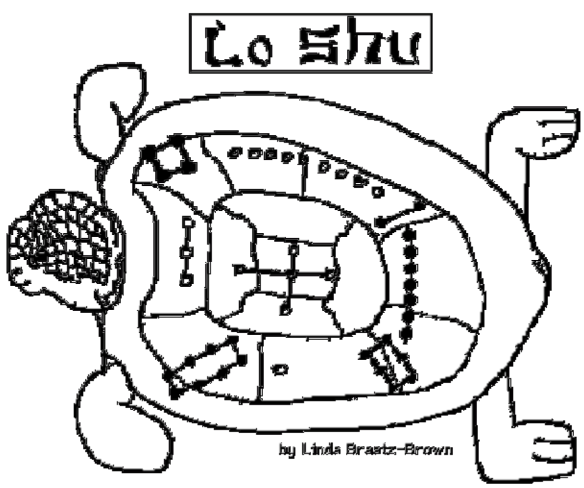
To proof the definition, we need to add numbers from each row, column and across of the square above. For example if we add 16 to 3 to 2 to 13, we will get

34. The same situation is with numbers across. The sum in this square is always 34.



The proof.

According to the Chinese legend, one magic square was shown to the emperor called Lo Shu on the turtle shell (about 2800 b.c.). Lo Shu thought that the numbers on the shell may be the hint how to protect the country against a cataclysm. This strange way of the magic square appearance can make it magical. But the mathematical magic is in the identical sums and that in the square there are digits from 1 to 9.



I asked myself a question if I can make other magic squares dimension 3 completed with the same numbers? I found a statement which says that there are 8 magic squares with the same numbers.
To proof it we need to chose any magic square.

8	1	6
3	5	7
4	9	2

.

To make new settings of the same numbers we need to swap a column with a row. The two colors: red and blue show the changes.

8	1	6
3	5	7
4	9	2

4	3	8
9	5	1
2	7	6

6	7	2
1	5	9
8	3	4

2	9	4
7	5	3
6	1	8

Next layouts are made when in already done we swap side columns with each other. In the first square we can see that blue and red digits just swap places.

6	1	8
7	5	3
2	9	4

8	3	4
1	5	9
6	7	2

4	9	2
3	5	7
8	1	6

2	7	6
9	5	1
4	3	8

Because of these squares we prooved that there are 8 magic squares made by using the same numbers. If we wanted to swap the numbers in squares once again, we will have the same figure as we had before.

Next is a Latin Square which is a distribution of numbers {1, 2, 3,...} on a square chart that in every row and every column is every number from 1 to n where n is the number of rows and columns.

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Example of a Latin Square dimension 5, completed with digits from 1 to 5.

I made the square above the easiest way. It means that I completed first row with numbers from 1 to 5 and in every row under, I moved every digit in one place, that across the square is always 1. With that definition we can make the Latin square of any n.
Can we ask a simillar question as about the magic squares? So, can we make other Latin squares dimension 3 using the same numbers?

First, if we put across Latin squares the same numbers (first 1, then 2, next 3) we will get.

1	2	3
3	1	2
2	3	1

1	3	2
2	1	3
3	2	1

2	3	1
1	2	3
3	1	2

2	1	3
3	2	1
1	3	2

3	1	2
2	3	1
1	2	3

3	2	1
1	3	2
2	1	3

I marked the digits across just to show how it works. After that I needed to complete left places with the Latin square condition. Now we have 6 different Latin squares from the same numbers.

To find new settings let's look at the front row. First, in the left top corner we put 1 and complete it with the rest of numbers, then we put 2 and 3. That's how we get these squares:

1	2	3
2	3	1
3	1	2

1	3	2
3	2	1
2	1	3

2	3	1
3	1	2
1	2	3

2	1	3
1	3	2
3	2	1

3	2	1
2	1	3
1	3	2

3	1	2
1	2	3
2	3	1

That way we proofed that there are no more Latin squares dimension 3 made from the same numbers.

Next topic is a Greek-Latin square which is a distribution of elements of files S={A, B, C, D, E,...} and T={α, β, γ, δ, ε,…} that in any row and any column there aren't the same elements from any of these files (they make Latin squares) and done elements (s, т) are every different.

Aα	**Bβ**	**Cγ**
Bγ	**Cα**	**Aβ**
Cβ	**Aγ**	**Bα**

Examle of a Greek-latin square dimension 3.

Let's focus on how it works. First we need to have one Latin square completed with Latin letters.

A	B	C
B	C	A
C	A	B

Next we have to create the second Latin square using Greek letters.

α	β	γ
γ	α	β
β	γ	α

If we put these two squares together we will get Greek-Latin square like in the first example.

Aα	Bβ	Cγ
Bγ	Cα	Aβ
Cβ	Aγ	Bα

Next I am going to work on Sudoku. This is a square chart dimension 9 divided into nine squares dimension 3. We need to complete it with numbers from 1 to 9 that in every row, every column and every little square there will be one digit from 1 to 9. We usually know a few numbers. Then we should complete the others.

7	4	8	**6**	**5**	9	**3**	1	2
2	6	**5**	4	1	3	9	**7**	8
3	9	**1**	8	7	2	**6**	4	**5**
4	**5**	9	1	3	6	**8**	2	**7**
8	2	6	7	4	5	1	3	9
1	3	**7**	9	2	8	5	**6**	4
9	8	**3**	2	**6**	4	**7**	5	1
6	1	2	**5**	9	7	**4**	8	3
5	7	**4**	3	**8**	1	2	9	6

Example of a Sudoku.

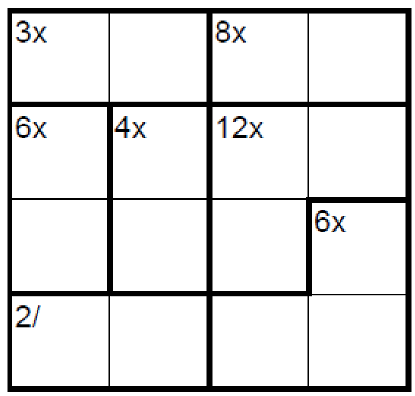
The red digits are the ones which are given to us before completing this chart. Next, we have to write other numbers from 1 to 9 to have Sudoku done.
Now we are going to concentrate on KenKen Square which is logical-arithmetical game discovered by Japanese teacher Tetsuya Miyamato. Literally the name means "clever square" in Japanese. KenKen square has to be dimension from 3 to 9. You complete it with numbers from 1 to n that they will make Latin squares and that all the conditions are met.

11+ **5**	2/ **6**	**3**	20x **4**	6x **1**	**2**
6	3- **1**	**4**	**5**	3/ **2**	**3**
240x **4**	**5**	6x **2**	**3**	**6**	**1**
3	**4**	6x **1**	7+ **2**	30x **5**	**6**
6x **2**	**3**	**6**	**1**	**4**	9+ **5**
8+ **1**	**2**	**5**	2/ **6**	**3**	**4**

Example of a KenKen square dimension 6 ,comleted with numbers from 1 to 6.

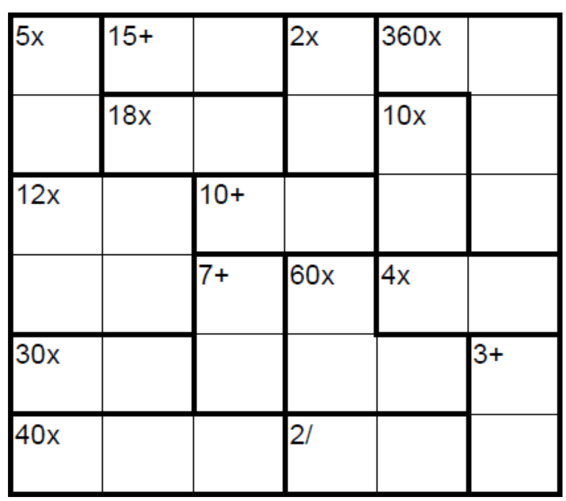
If we can see „11+" that means that we should add the numbers in the bold places to get 11 as a solution.
If we have „2/" we need to divide the numbers to get 2 as a solution.
If we see „6x" that means we have to multiply this numbers to get 6.
I have made two KenKen squares which can be done by yourself.

Easy:



KenKen square dimension 4.

Hard:



At the end I want to say that it is just the beginning of my math road. While making this Project I have learnt many new things. I have done every square by myself. I had lots of fun preparing it and this can be known as magical. ☺

STRESZCZENIE PRACY

W trakcie mojej prezentacji będę mówić o kwadratach magicznych. Najwyraźniej pierwsze pojawienie się kwadratu magicznego wymiaru 3 jest związane z chińską legendą o żółwiu Lo Shu, który wyłonił się z Żółtej Rzeki z wyjątkowym wzorem na jego skorupie.
Będę w skrócie prezentować tę legendę. Później przejdę do renesansu i opowiem o magicznym kwadracie wynalezionym przez niemieckiego artystę i naukowca Albrechta Durera. Następnie pokażę, jak z magicznych kwadratów powstały łamigłówki. Właściwie to było motywacją do moich poszukiwań. Zagadki to moja ulubiona dziedzina matematyki i dlatego chciałabym się podzielić moją pasją w tej prezentacji.

ON PECULIAR PROPERTIES OF THE PASCAL TRIANGLE

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ABSTRACT

The Pascal triangle is a triangle consisting of numbers arranged in certain pattern. It is named after French philosopher and mathematician Blaise Pascal (1623-1662). The k-th number in the n-th row of the triangle we denote for simplicity and in order to honor Pascal by P(n,k). This number is exactly the binomial coefficient

$$P(n,k) = \binom{n}{k}$$

which counts how many subsets of k elements can be chosen from a set of n elements. There is the well-known formula

$$P(n,k) = (P(n-1,k-1) + P(n-1,k),$$

which allows to build the triangle in a recursive way. The k-th entrance in the n-th row is the sum of the entrances (k-1) and k in the (n-1)-st row.
Playing with triangle and browsing through the literature, we have discovered that there is a big number of other very interesting properties of Pascal's triangle.

Let's start. A Pascal's triangle is a matrix made of numbers arranged this way: there is a 1 on the top and each number is a sum of two elements placed above it. Guess the missing number under 4 and 6.

How is the triangle made?

Pascal's triangle is a matrix made of numbers arranged this way: there is a 1 on the top.
Any particular number is the sum of two numbers placed above it.

					0		1		0					
				0		1		1		0				
			0		1		2		1		0			
		0		1		3		3		1		0		
	0		1		4		6		4		1		0	
0		1		5				10		5		1		0

How do we find numbers in the Pascal's Triangle? We use the Newton's symbol. See how it works below:

Newton's Symbol

To understand the symbol, we have to know what a factorial is.

$n! = 1\times2\times3\times\ldots\times(n-1)\times n$

Expansion: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$\boxed{\binom{n}{k} = C_n^k}$!

Example: $\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = 10$

Usage: Combinatorial analysis and the binomial theorem

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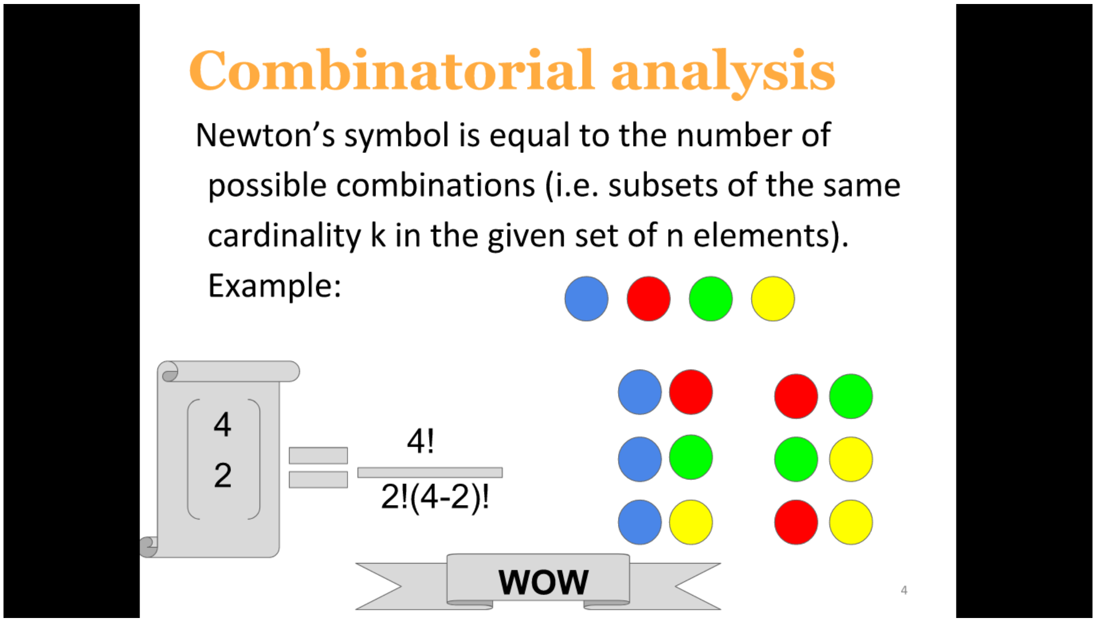
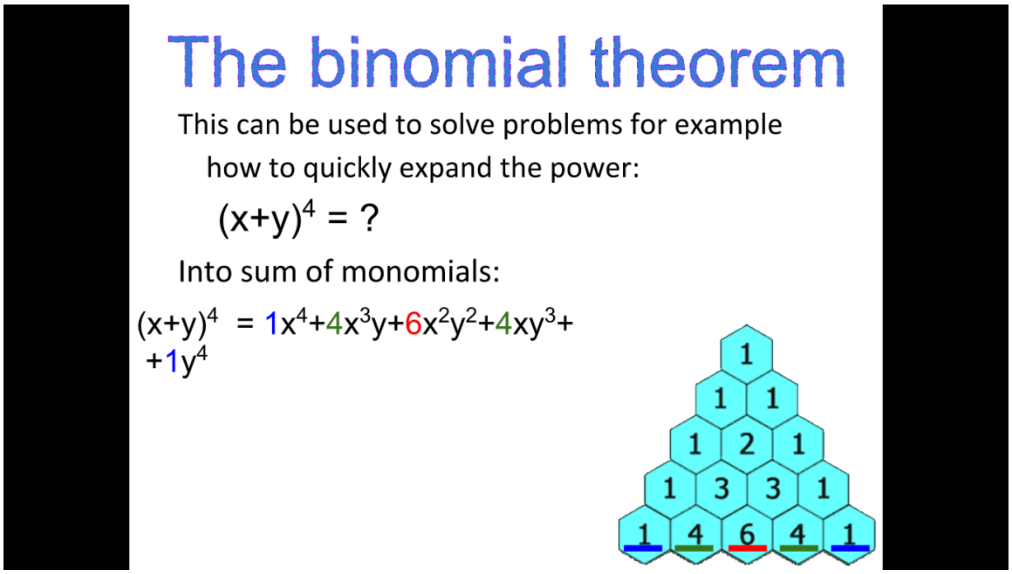
This example shows us a ‘10’. This is because we look at the 5th row (starting from zero, so the ‘1’ on the top counts as a row 0), and then we look at the second number in this row (also counting from zero), which is 10.
Here is another usage of the symbol:

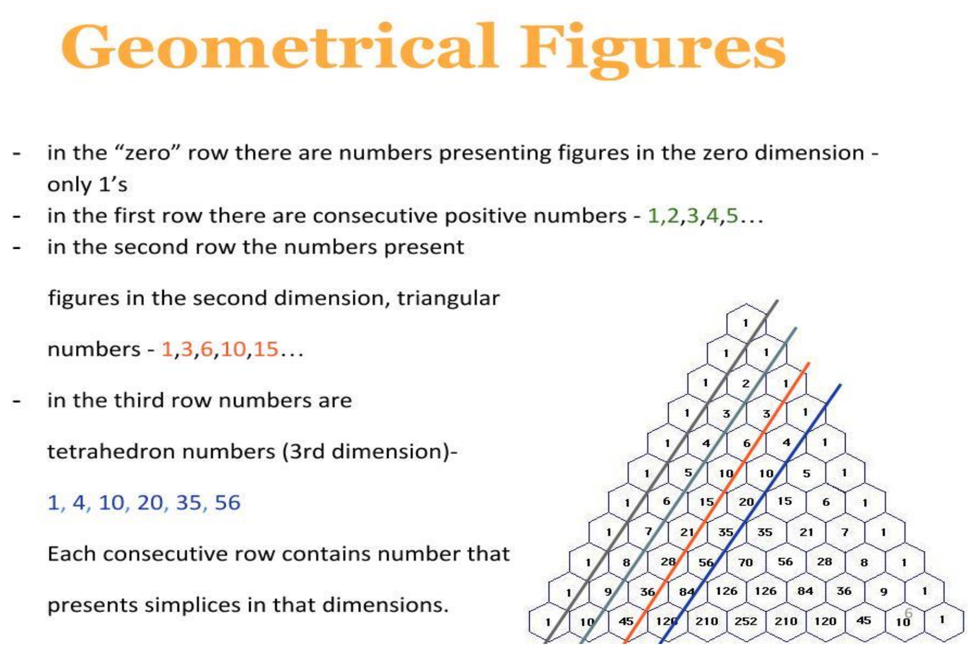


Let’s imagine you have four balls in different colours (like on the picture above). The goal is to count how many combinations there are. Of course, we can make pairs, but the faster way is to use the Newton’s symbol and just count it. So, we take 4 as the n (because we have four balls) and k as 2 (because we make pairs).

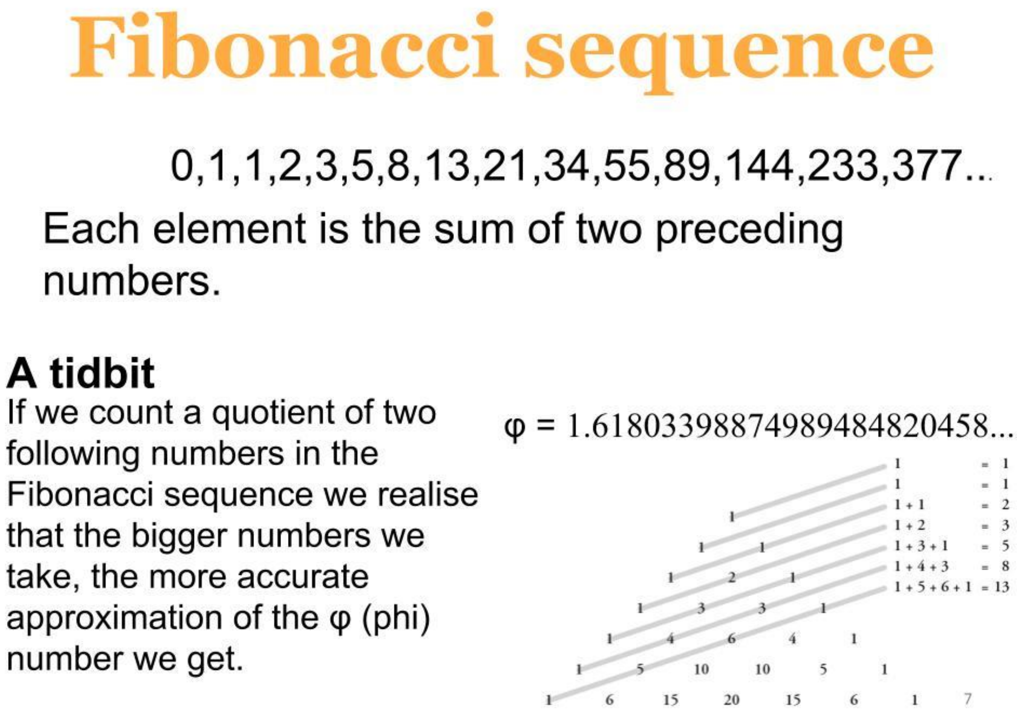
The next thing connected to the triangle is the binomial theorem. We use the Pascal’s Triangle to ease problems like this one below. To do it we have to check the row that equals the power. See below:



In Pascal’s Triangle we can also see rows that contain particular numbers and these represent simplices numbers size n in particular dimensions.



There is a Fibonacci sequence appearing in Pascal’s Triangle too. Sum of numbers in rows gives us consecutive elements of Fibonacci sequence.



There are two main properties of Pascal's triangle: the first one is that sum of numbers in each row is 2 times bigger than in previous one so they create consecutive powers of two.

Breathtaking feature of Pascal's triangle

If we sum every row of pascal's triangle,
we get consecutive powers of two.

$$
\begin{array}{ccccccccccc l}
 & & & & & 1 & & & & & & =1=2^0=S_0 \\
 & & & & 1 & & 1 & & & & & =2=2^1=S_1 \\
 & & & 1 & & 2 & & 1 & & & & =4=2^2=S_2 \\
 & & 1 & & 3 & & 3 & & 1 & & & =8=2^3=S_3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & =16=2^4=S_4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & =32=2^5=S_5
\end{array}
$$

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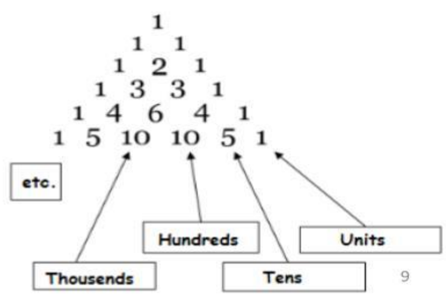
So, we can also create row-numbers by summing numbers in every row, but this time multiplying them by powers of 10 going from right. For example 1 and 1 will be 11, 1, 2 and 1 will be 121. We can see that these numbers create consecutive powers of 11.

And another one

Let's create a Row-number (R_n). A Row-number is created by connecting numbers in rows. Now we can see that the Row-numbers are the powers of 11.

$R_0 = 1$
$R_1 = 1 \times 11 = 11$
$R_2 = 11 \times 11 = 121$
$R_3 = 121 \times 11 = 1331$
$R_4 = 1331 \times 11 = 14641$
$R_5 = 14641 \times 11 = 161051$



You probably already know Pascal's Triangle, it was exciting in like 17th century, now it's not cool anymore. That's why we invented Johnny's Triangles! We can create them from as many numbers above as we want (that's L and it must be nonnegative and integer). Here is the example for Johnny's Triangle L=3.

Johnny's Triangle (L=3)®

			0	1	0			
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1

Let's define this triangle as a Johnny's triangle L=3 (whereas the Pascal's triangle is L=2). L=3 means that a particular number is a sum of 3 numbers placed above it.

Actually this is completely useless and we didn't discover anything interesting.We just did maths!

Here we can see some properties of Johnny's Triangle (L=3).

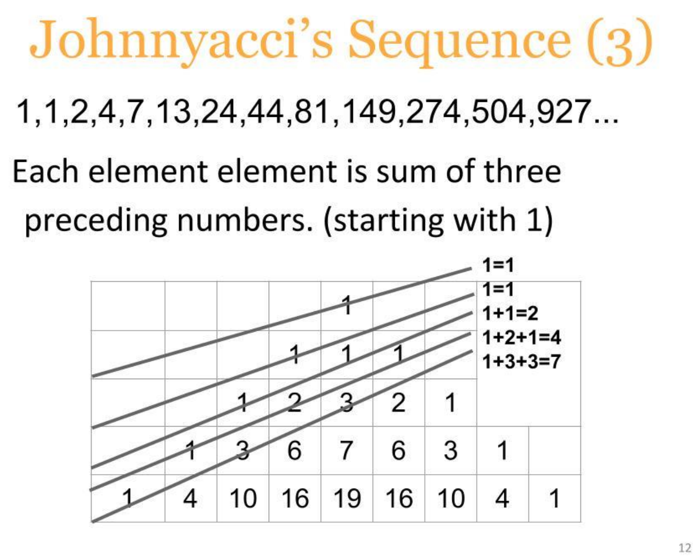
Johnny's Triangle (L=3)

In this triangle when we create number adding 3 numbers above it we can see that sum of digits in every next row is 3 times bigger, and row-numbers are 111 times bigger.

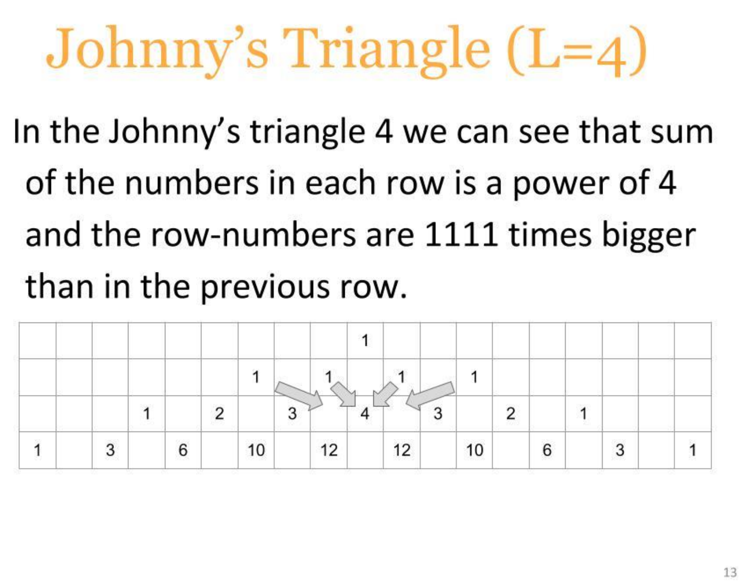
				1				
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1

11

When the normal Fibonacci sequence is made of numbers that are sum of 2 previous elements, numbers in Johnnyacci's sequence are sum of L previous elements. Here's the example of Johnnyacci's Sequence L=3.



We can also create another Johnny's Triangle (L=4). Each number will be a sum of two elements above it.

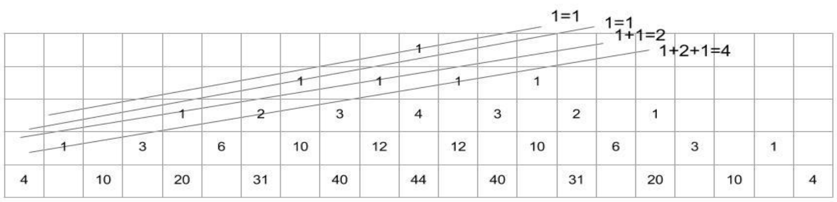


As with the Johnny's Triangle, we can create the next Johnnyacci's sequence (4), in which each element is a sum of four preceding elements.

Johnnyacci's sequence (4)

1,1,2,4,8,15,29,56,108,208,401,773,1490

Each element is a sum of four preceding numbers.



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Johnny's Triangle

These properties appear in every next Johnny's triangle but sum of digits in every row will be a L to power of number of the row.

$$S_n = L^n$$

And row numbers will be: 111 (L=3), 1111(L=4), 11111(L=5) times bigger, so

$$R_{L,n+1} = R_{L,n} \times \sum_{t=0}^{L} 10^t$$

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Also, we can realise that Johnnyacci sequences appear in all Johnny's Triangles. The only condition is that L have to be equal both in the sequence and the triangle. For example, Johnnyacci sequence (4) appears in the Johnny's Triangle (L=4).

SUPERPOWERS DEBUNKED USING SCIENCE

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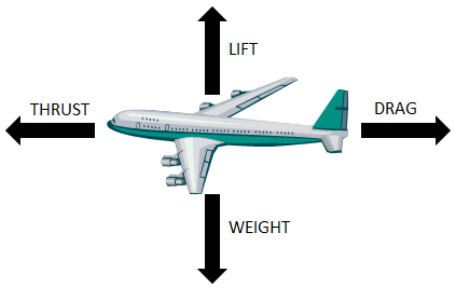
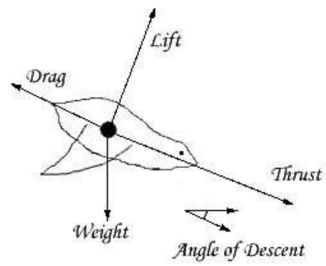
ABSTRACT

We often ponder about the possibility of having a superpower. Wouldn't having the ability to fly or super strength make our lives a whole lot easier? On a first thought, yes it would. Just think about it, why wait in traffic when you can just fly soaring through the sky to get to your destination or being invisible and using this ability to enter any place on earth without repercussions. It might sound good on paper but once you start adding logic to the mix, the whole notion of superpowers actually becomes an inconvenience without any real benefits. In this paper, we are going to examine the science behind three superpowers. These are super senses, the ability to fly and invisibility. By the term "super senses", we mean that someone is able to hear, see, smell, and taste in a wider spectrum that normal people can, but it's surely not as dreamy as it sounds. Invisibility, which is pretty self-explanatory, has a serious prize that comes with it, blindness. Finally being able to fly is one of the primary powers, but the sky hides many dangers.

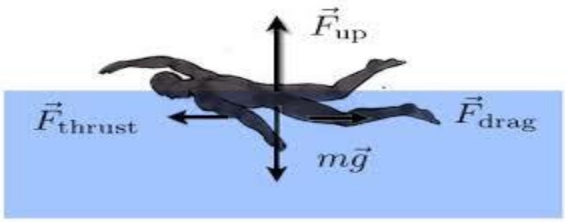
THE ABILITY TO FLY APPLIED TO THE REAL WORLD

The ability to fly is defined as the process in which an object moves above the ground. Flying in the real world can only be achieved in two ways, mechanical flight also known as aviation (planes, helicopters etc) and animal flight also known as gliding(most species of birds and insects).In both gliding and flying the same exact forces exist.





So how would an average human be able to fly? Would he grow wings? No because that wouldn't make him human but an evolved version with human characteristics. Would he use a jetpack or another kind of mechanical flying aid? No because that wouldn't be considered a superpower? Then how would humans be able to fly? Most superheroes in comics and movies achieve this by levitating so we would have to give the average human the ability to levitate so he could fly.
Now that we figured out in under which circumstances we would be able to fly it's time to get into the more technical aspect. First of all how fast would we go? While a first guess would be very fast due to the fact that air travel is faster we have to remember that this is achieved due to wings. A second guess would be around the average human running speed which is about 24 km/h, while the fastest recorded speed is about double and while very impressive it is still about the speed of a slow moving car. So we would be able to fly in about 24-48 km/h? No because we are forgetting a very important detail, that forward motion is achieved by our feet hitting the ground and the ground reacting on us by propelling us forwards(Newton's third law of motion states that for every action there is an equal and opposite reaction) and in the air the particles are a lot less dense than in the ground which would lead to a slower movement speed similar to that of swimming similar to how astronauts move in low gravity conditions.



The average swimming speed is about 5 km/h while the fastest recorded speed is about 8km/h. So flying would essentially be swimming in the air.
But that is just one way of debunking flight Even if we were able to fly and do so at a solid pace we would experience diver's disease which happens when deep sea divers rise suddenly to the surface but also when flying in an unpressurised aircraft or when moving in space.It is caused by a reduction in ambient pressure (surrounding pressure) that results in the formation of bubbles of inert gases

within tissues of the body, which lead to a plethora of symptoms. So even before you start your flight you would most likely have dropped to the ground and in the best case scenario become hospitalized.

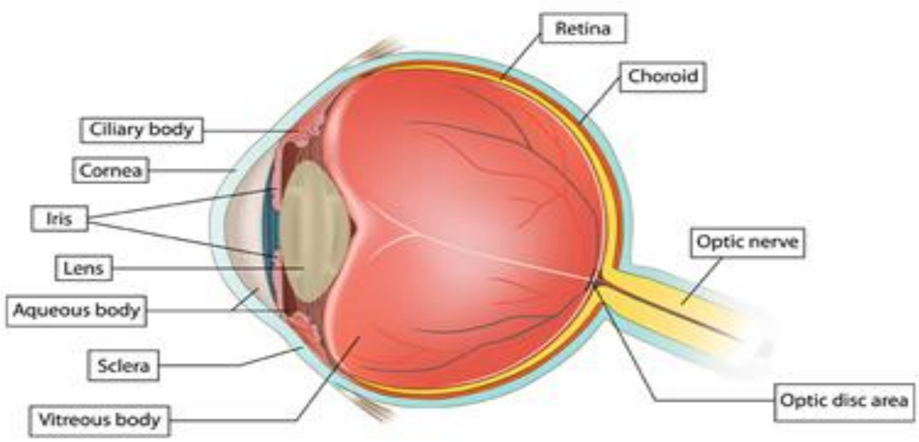
Symptoms by frequency

Symptoms	Frequency
local joint pain	89%
arm symptoms	70%
leg symptoms	30%
dizziness	5.3%
paralysis	2.3%
shortness of breath	1.6%
extreme fatigue	1.3%
collapse/unconsciousness	0.5%

Another they that we would be able to debunk flight, is that we would still be subject to air temperatures that steadily drop due to fact that the atmospheric pressure becomes lower at higher altitudes because the Earth's atmosphere feels less pressure the higher up you go. So as the gas in the atmosphere rises it feels less pressure and the particles become less dense and they generate less heat because they use less energy.

INVISIBILITY

Light strikes the retina and initiates a cascade of chemical and electrical events that ultimately trigger nerve impulses. These are sent to various visual centres of the brain through the fibres of the optic nerve.



This picture shows an ideal eye with perfect focus. All the rays of light traveling through the eye focus as a single image on the retina. When someone is invisible, the light doesn't bounce off, but instead it goes right through you or around you. Except from the scientistic problems, we can use our minds to see that the logical problems are quite visible. It's natural to assume that being invisible, doesn't make your clothes invisible too. So, if you want to go unnoticed you have to be naked. That leaves you exposed to the weather, from the wind to the snow. So if you live in New York for example, in the winter it can get -10 Celsius degrees. Another problem is that you have to get used to be unnoticed. For example crossing the street is deathlier than fighting a villain. We are used to the cars noticing us and stopping, slowing down or at least honking at us to warn us. An invisible person has to be careful, or he will get run over by a car or be stepped on. At last in some circumstances other people can see you even thought you are invisible. For example, if it's raining or snowing, the outline of your body will be visible for everyone to see. The same is for dust or spilled coffee.

SUPERHEARING

Now, we are going to analyze the super- power of super hearing:
But what does it mean, to have super- hearing? That's a rather reasonable question. As it's a term first used in comics, referring to super- heroes, to have super- hearing means to be able to listen to frequencies normal people cannot and hearing everything going on around you. Literally, you can hear anything happening in a 10 block radius. But first we shall define :Frequency: Frequency is the number of occurrences of a repeating event per unit time Audio Frequency:An audio frequency (abbreviation: AF) or audible frequency is characterized as a periodic vibration whose frequency is audible to the average human. The SI unit of audio frequency is the hertz (Hz). It is the property of sound that most determines pitch.

Frequency (Hz)	Octave	Description
16 to 32 Hz	1st	The lower human threshold of hearing, and the lowest pedal notes of a pipe organ.
32 to 512 Hz	2nd to 5th	Rhythm frequencies, where the lower and upper bass notes lie.
512 to 2048 Hz	6th to 7th	Defines human speech intelligibility, gives a horn-like or tinny quality to sound.
2048 to 8192 Hz	8th to 9th	Gives presence to speech, where labial sounds lie.
8192 to 16384 Hz	10th	Brilliance, the sounds of bells and the ringing of cymbals and sibilance in speech.
16384 to 32768 Hz	11th	Beyond Brilliance, nebulous sounds approaching and just passing the upper human threshold of hearing

An average human is able to hear sounds that vary from 20 to 20,000 Hz. Nevertheless, sensitivity in frequencies, especially 16 kHz- 20 kHz), differs between individuals. Let me show you what this means in practice. We are about to hear frequencies starting from 20 Hz rising till they reach 20 kHz. Euroscience community, are you sure you are ready for this? Biologically speaking, the majority of you must have felt pain after hearing the piercing sound of high frequencies. The table below represents the hearing spectrum some animals have, including human:

Animal	Hearing range in Hertz
Humans	20 – 20,000
Bats	2000 – 110,000
Elephant	16 – 12,000
Fur Seal	800 – 50,000
Beluga Whale	1000 – 123,000
Sea Lion	450 – 50,000
Harp Seal	950 – 65,000
Harbor Porpoise	550 – 105,000
Killer Whale	800 – 13,500
Bottlenose Dolphin	90 – 105,000
Porpoise	75 – 150,000
Dog	67 – 45,000
Cat	45 – 64,000
Rat	200 – 76,000
Opossum	500 – 64,000
Chicken	125 – 2,000
Parakeet	200 – 8,500
Horse	55 – 33,500

However, let's assume that someone is able to hear at a wider spectrum than a normal person. Humans do not have the ability to turn off their senses, and if they did, then senses would be useless. How would he hear a crime happening 10 miles away from his position and not the 7,000 televisions playing in between? Well, the answer is: he wouldn't. As a result, he couldn't hear anything very clearly. To top it all, he wouldn't even be able to listen to his own thoughts, or sleep. Which can actually even lead to death. As Bill Budd, a lecturer and scientist at the University of Newcastle states: "Hearing isn't like other senses; if a light is too bright we can always close our eyes or turn away, but our hearing is always 'on', even when we are asleep, and super-sensitivity would be quite impairing. We would be surrounded by such a cacophony of sound we wouldn't hear anything very well."

SOURCES:

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MATHEMAGIC

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ABSTRACT

Thinking about cards the first thing that comes to mind is games and gambling. Well, apart from that there is a whole other world that lies beneath these unexplored waters. This world is in correlation with logic and mathematical thinking. In our paper we will refer to the history of playing cards and all the mathematics behind mind boggling tricks. We will start by mentioning the origins of the names of all different suits and figures. Furthermore we are going to discuss the theory of possibilities behind cards and their never ending permutations. The core of our paper will be the detailed mathematical explanation of two magic tricks, the 27 card trick and the 44 card trick. We will attempt to demolish the illusion of magic and prove that mathematics is the reason hidden behind the magic curtain. Mathematics is hidden everywhere, in the most unexpected places in our daily lives. The most important thing in order to critical thinking is to uncover and understand the omnipresence of mathematics. Shall we deal?

HISTORY

Playing cards were first invented in China as the Chinese population was the one who invented paper and printing. Initially they were made by hand and because of that they were expensive and accessible only to the upper class. By the end of the 14th century there was a stir throughout Europe. There existed a huge variety of games and decks of cards were easily accessibleto the public. Gambling was also popular at that time. This made the church and the state control the sale and use of playing cards. It must be noted that the church was opposed to card games and sometimes it was forbidden. People were informed that playing cards were evil and sinister.

Cards had also other uses for example in the 18th century in the Netherlands mothers used the back of a cards to write a message which would identify an abandoned child. If the card was torn in half it meant that the mother may someday return with the other half of the card to get her child back, if the card though was whole it meant that the mother had completely abandoned her child.

As for the suit marks there are a lot of theories surrounding them. The majority believes that it is interplay between words, shapes and concepts. Others now believe that the images in the cards are the work of secret societies. For example the four suits may refer to the four seasons, the 52 cards may represent each week of the year and the 13 cards per suit may refer to the cycles of the moon. The truth is that there is no proof which confirms or disapproves any of the given theories so we cannot actually rely on them.

Something though that we know is how playing cards reached Europe and how the final design of the suit marks was created. Playing cards and their suits reached Europe from the Mameluke Empire of Egypt. The Mameluke suits where goblets, coins, Swords and polo-sticks. Because Polo sticks were not known to Europe they transferred it into batons and stave wits along with swords, cups and coins which are still traditional to Italian and Spanish cards.

Then in the 17th century German card makers experimented on different suits based on the Italian ones but they ended up to acorns , leaves, hearts and bells or else hawk-belles which remain in use. In 1480 the French simplified the German shapes into trèfle, pique, hearts and paving tiles or carreau. The final step though was made from the English card makers, who used these shapes but changed their names. Pique was renamed into spade and it was similar to swords and clubs which are what this Spanish Stave looks like. Diamond was not only similar to paving tiles which were Carreau but also to the older sweet of coins.

The final piece of information is about the history of the joker. Some argue that the word joker comes from euchre which is a card game which means coincidence, but the word joke was actually already in use. The joker card meant the one whose jokes about the concept behind that card was completely different.

Around the 1860s American euchre players made up some new rules which required an extra Card which they referred to as the best bowler which was the highest card available. The American card printers fairly quickly included this extra card in their decks and so did the British ones by the 1880s. It was not long until the best bower was called joker or else jolly joker. They had a unique design that contained a company's brand image. These Chester became even more popular when the Joker title was universally adapted.

PROBABILITIES AND PERMUTATIONS

We have already examined the origins of playing cards, but now let us proceed to explore the permutations and possibilities behind them. So what is the possibility of us drawing a specific suit of cards from the deck? For example, what is the probability in a standard deck of 52 cards to randomly select one card that is a spade? It is known that a playing card can be either a diamond, spade, heart or club and the 52 cards are divided in 4 groups each of 13 cards. We define the possibility

A: the card that we choose is a spade

According to the definition of probability we need to calculate the number of all possible results, it is clear that all possible results when a card is chosen randomly are 52.

Then we have to calculate the number of favorable results which is how many of the cards are spades. As we saw previously there are 13 cards that are spades so according to the definition of probability in a deck of 52 cards

$P(A) = 13/52 = 1/4 = 0{,}25$

We have 25% chance that the card we picked is a spade.

Now let us see what is the number of different ways a deck of 52 cards can be arranged. The top card of the deck could be any of those 52, so there are 52 different possibilities, for the second card from the top there are 51 different possibilities, as we have already removed one card from the deck, for the third it is 50, the fourth 49 and so on. So if we multiply all those numbers together we will get how many unique ways there are to arrange 52 cards.

52*51*50*49…*1=52!

This gives as 52!(factorial). A factorial is the product of a given positive integer multiplied by all lesser positive integers. n! shows us the permutations for n different object, n!=n*(n-1)*(n-2)*…*1. So a deck of 52 cards has 52! different arrangements or permutations.

52! is an immensely large number. It is equal to $8.065817517*10^{67}$. This means that when you shuffle a deck of cards really well you are most likely to arrange it in a way that has never existed before. There are more ways to arrange a deck of cards than there are atoms on earth. If you created a new arrangement of those cards each second ever since the universe was created, 13.7 billion years ago, today you would not have even reached half of those permutations.

Analytics professional, Scott Czepiel has proposed an excellent example of demonstrating the size of 52! :

Let us say that you set a timer to count down 52! seconds, then imagine that you are standing on the equator. You are going to walk around it, but you can only take a step every billion years. When you reach full circle and return to the point from where you started, remove one drop from the Pacific Ocean. Repeat the same process until you have emptied the Pacific Ocean. When that is done set down a piece of paper, refill the ocean and start all over again. When the stack of papers reaches the sun look at you timer, you will notice that the time has changed very little. Do the same thing 1,000 times and you will be 1/3 of the way done.

In order to pass the time that is left take a deck of cards and deal yourself five cards every billion years. When you get a royal flush buy a lottery ticket and every time the ticket wins the jackpot throw a grain of sand in the Grand Canyon. Once the Grand Canyon is full remove 28.3g of sand from Mt. Everest, empty the canyon and start all over again. When you have leveled Mt. Everest look at the timer. You are still not done. Repeat this process 256 times and then and only then will the timer have reached zero.

Of course this example is hypothetical and surreal as it is not feasible; however it perfectly demonstrates the gigantic number that is 52!

THE 27 CARD TRICK

The last part of our paper is the presentation and explanation of two mathematics-based magic tricks. The first trick we are going to investigate is called the 27 card trick. The name itself gives away the information that this trick requires 27 cards which we are going to then separate to three columns. Those two numbers 3 and 27 are really important in order to later understand the mathematics behind the trick.

The participant from the audience selects a card at random and names a number from 1 to 27. The magician separates the cards in 3 columns 3 times and asks the member of the audience to remember where their card was. In the end, the magician is able to place the chosen card in a specific position in the deck of 27 cards. For example if the participant chooses the 4 of clubs and says the number 14, then the magician removes 13 cards from the top and the 14^{th} card is the 4 of clubs.

Let us now examine the mathematics behind this impressive trick.

The trick uses the Ternary numeral system , which is also called "base three" and has three as its base. The digits are called trits and one trit is equivalent to log 23 (about 1,58) .

For instance:

0+0=0

0+1 or 1+0 equals 1

1+1=2

But, 2+1=10 because we carry over the 1

2+2=11 and we carry over the 1

The orders of magnitude of base three are: 1,3,27,81,243,729,2187,6561… where the lowest order number is itself one time and the next number represents itself 3 times. So we have the number 42 in base 10 decimal which in base 3 is the number 1120. 42 is analyzed to:

i. No 1s
ii. Two 3s
iii. One 9
iv. One 27

Which when added gives us 42. 2*3+9+27=42

Let us now return to the explanation of the trick.

If the participant says the number 23, for example, then the magician has to transform the previous number (22) to a base three number, in order to get the card to the 23rd position. So 22 has one 1, one 3 and two 9s.

The 1st time the magician compiles the three columns into one pack of cards they chose how many ones they will have, the 2ndtime the threes and 3rd time the nines. If you have none of one of the numbers you put the column with the card on top, if you have one you put it in the middle and if two in the bottom. This is actually moving the card in the position you want it to be at the end. So 22, which is needed to place the card in the 23rd position, would be: bottom, middle, middle.

2: bottom 1: middle 0: top

3rd Deal	2nd Deal	1st Deal	number
0	0	0	1
		1	2
		2	3
	1	0	4
		1	5
		2	6
	2	0	7
		1	8
		2	9
1	0	0	10
		1	11
		2	12
	1	0	13
		1	14
		2	15
	2	0	16
		1	17
		2	18
2	0	0	19
		1	20
		2	21
	1	0	22
		1	23
		2	24
	2	0	25
		1	26
		2	27

THE 44 CARD TRICK

First you take 9 cards out of the deck and you let the volunteer choose one and put it on top of the 9 cards. This reveals that the card must always be in the 44th place. Then, you open 4 piles of cards and as you open a card you say "10,9,8,7,6,5…" all the way to 1. If for example you say 5 and it is 5 then the magician stops and moves on to the newt pile. If you don't get a number you put a closed card on top and you must remember that the figures count for 10. After you have opened all the piles you count the numbers that are on top of the piles and you get a sum "n". The card that we are searching for will be in the nth place. But why does this happen? 44 divided by 4 is 11. When you open the piles, if you stop for example at 8 you will have opened 3 cards, so 8+3(cards)=11. If you stop at 5 you will have opened 6 cards so 5+6=11. If you stop at no card and you put a closed card on top you have 0 and you will have opened 11 cards so 0+11=11. So whatever you do, whenever you stop, you will always have 11 and you have 4 piles so 11+11+11+11=44 which is the position of your card.

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THE UNBEARABLE LIGHTNESS OF GRAVITY

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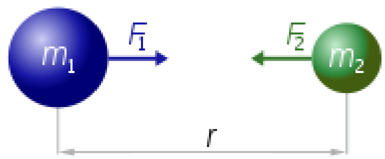
ABSTRACT

Gravity or gravitation is a force that exists among all material objects in the universe. One of the main reasons that contribute to our existence is gravity. So someone could say that gravity is everywhere, it pulls on everything and it is something we can never escape from. In this paper we will explore what would happen if gravity suddenly switched off. What would unexpectedly happen to human beings, animals and the planet itself? How would the solar system look like? The world itself would not be the same without the existence of gravity let alone planet earth. In our research we will examine the training of Astronauts and Cosmonauts who lived for some months on ISS. As we can never escape from gravity on Earth there are some extraordinary ways for the astronauts training. Their experience on an environment without gravity will be our starting point in this attempt to describe how human life would be without the existence of gravity.

Gravity or gravitation is a force that exists among all material objects in the universe. One of the main reasons that contribute to our existence is gravity. Gravity is everywhere, it pulls on everything and it is something we can never escape from.

Newton's law of universal gravitation, which was published in 1687, states that two objects are attracted by the force F of gravity. This force is directly dependent upon the masses m_1, m_2 of both objects and inversely proportional to the square of the distance r that separates them.



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

Albert Einstein described in **General relativity,** which was published in 1915, that gravity is not really a force but the result of the curvature of the four dimensions of spacetime. All the objects are bending the spacetime around them. The bigger the object is, the more the spacetime is bending
Gravity holds together the galaxies, makes the planets moving in orbit, gives to stars and planets their spherical shape, captures gasses in a planet's atmosphere and water in lakes and oceans. Gravity makes life possible in our planet. On the other hand, gravity was responsible for the extinction of dinosaurs, pulling meteors to crash into Earth 60 million years ago.

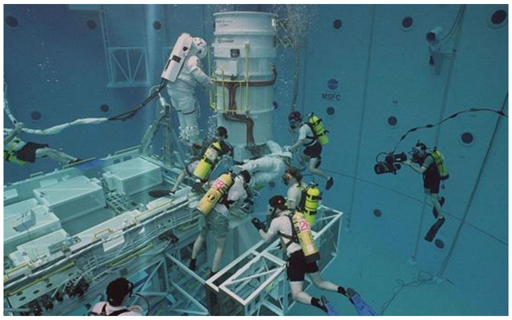
TURN THE GRAVITY OFF

If our planet's gravity suddenly turned off, everything it's not stuck in ground would have no reason to stay down. People, cars, animals, furniture would start floating. The Earth's atmosphere would also float off into space. The molecules of the atmosphere, due to their thermal energy, will start moving away from our planet. Earth's oceans, lakes, and rivers would also depart. Fishes, submarines, boats and icebergs will float in enormous bubbles of water high in the sky. Even Earth itself would break apart into pieces and due to its rotation would float off into space. A lack of gravity would make Moon float off into space, since only Earth's gravitational force is keeping the Moon in orbit.

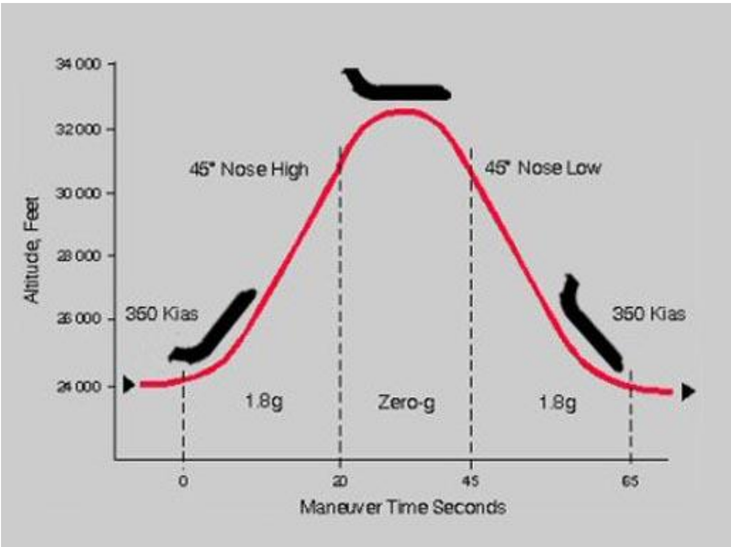
ASTRONAUTS TRAINING

Life is impossible without gravity. So how the astronauts and the cosmonauts manage to survey in space?
There are two methods to train astronauts for living in microgravity:
Swimming pool: Microgravity is being simulated in a pool by wearing a suit to an astronaut and adjusting that suit's weight until the astronaut neither floats nor sinks in the water, making it neutrally buoyant. N.A.S.A. has built a swimming

pool 10 times as large as an Olympic pool, in Houston Texas, in an attempt to prepare astronauts for the experience of weightlessness.



Parabolic Flight: The ZERO-G aircraft gives the astronauts the sensation of weightlessness by following a parabolic flight path. Initially the aircraft climbs up with steep angle and high velocity, then it becomes horizontal and followed by high speed nose down maneuver. The sensation of weightlessness begins as the up-climbing plane starts to slow down and ends till the craft is pointing downward with an angle of 45 degrees. This aircraft is used to train astronauts in zero-g maneuvers, giving them about 25 seconds of weightlessness out of 65 seconds of flight in each parabola. During such training, the airplane typically flies about 40–60 parabolic maneuvers.



INTERNATIONAL SPACE STATION

The **International Space Station** (**ISS**) is an artificial satellite in an orbit of between 330 and 435 km from the surface of the Earth. The station has been occupied since November 2000. Living in ISS is not easy at all.

SLEEPING IN SPACE

Astronauts go to bed at a certain time and they are scheduled for eight hours of sleep. Space has no "up" or "down," but it does have microgravity. As a result, astronauts are weightless and can sleep in any orientation. However, they have to attach themselves so they don't float around and bump into something. They usually sleep in sleeping bags located in small crew cabins. Each crew cabin is just big enough for one person.

EXERCISING IN SPACE

Exercise is an important part of the daily routine for astronauts aboard the station to prevent bone and muscle loss. On average, astronauts exercise two hours per day.

EATING IN SPACE

Eating in microgravity is very different than eating on Earth.
Astronauts eat three meals a day: breakfast, lunch and dinner. They can choose from many types of foods such as fruits, nuts, peanut butter, chicken, beef, seafood, candy, brownies, etc. Available drinks include coffee, tea, orange juice, fruit punches and lemonade. An oven is provided in the space station to heat foods to the proper temperature but here are no refrigerators. Condiments, such as ketchup, mustard and mayonnaise, are provided. Salt and pepper are available but only in a liquid form. This is because astronauts can't sprinkle salt and pepper on their food in space. The salt and pepper would simply float away. There is a danger they could clog air vents, contaminate equipment or get stuck in an astronaut's eyes, mouth or nose. Space food comes in disposable packages. The food packaging is designed to be flexible and easier to use, as well as to maximize space when stowing or disposing of food containers.

CLEANING YOURSELF IN SPACE

Cleaning yourself in space can be difficult. Astronauts wash their hair, brush their teeth, shave and go to the bathroom. However, because of the microgravity environment, astronauts take care of themselves in different ways. Astronauts wash their hair with a "rinseless" shampoo that was originally developed for hospital patients who were unable to take a shower. Because of microgravity, the space station toilet is more complex than what people use on Earth. The astronauts have to position themselves on the toilet seat using leg restraints. The toilet basically works like a vacuum cleaner with fans that suck air and waste into the commode. Each astronaut has a personal urinal funnel that has to be attached to the hose's adapter. Fans suck air and urine through the funnel and the hose into the wastewater tank.

HEALTH IN SPACE

Space is a dangerous, unfriendly place. Isolated from family and friends, exposed to radiation that could increase your lifetime risk for cancer, a diet high in freeze-dried food, required daily exercise to keep your muscles and bones from deteriorating, a carefully scripted high-tempo work schedule, and confinement with three co-workers picked to travel with you by your boss.
Astronauts gain a couple of inches in height during long-term space flight, but lose vital muscle mass.
Radiation can penetrate living tissue and cause both short and long-term damage to the bone marrow stem cells which create the blood and immune systems
Radiation has also recently been linked to a higher incidence of cataracts in astronauts
Another effect of weightlessness takes place in human fluids. The body is made up of 60% water. Within a few moments of entering a microgravity environment, fluid is immediately re-distributed to the upper body resulting in bulging neck veins, puffy face and sinus and nasal congestion which can last throughout the duration of the trip and is very much like the symptoms of the common cold

TWINS STUDY

Spending a year in space affected former NASA astronaut Scott Kelly's body in subtle but potentially significant ways, new research suggests.
Scott Kelly flew the first-ever yearlong mission aboard the ISS. The goal of the project is to gauge the physiological and psychological impacts of long-duration spaceflight, to help prepare for crewed missions to Mars.
Scott Kelly's identical twin brother Mark — a former NASA astronaut who flew on four space shuttle missions, stayed on the ground during Scott's yearlong

flight, serving as an experimental control that would allow research teams to detect genetic changes that spaceflight induced in Scott

For example, one team found that the telomeres — the regions at the ends of chromosomes — in Scott Kelly's white blood cells got longer during the mission. Telomeres help protect chromosomes from deterioration, and they get shorter over the decades as people age.

Another research team found an apparent decrease in bone formation during the second half of Scott's space mission, and another group identified a slight decrease in cognitive ability (thinking speed and accuracy) shortly after he touched down.

Conclusion

Gravity is everywhere. Nothing can escape from it. All research till now shows that living without gravity is a challenge for Science. There is a lot to be done to have safe space trips in long destinations such as Mars. We must not forget that 15 times per day we may look at the sky and see the ISS heroes.

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AN EDUCATIONAL APPROACH TO L-SYSTEM FRACTALS THROUGH THE NEW TECNOLOGIES in PBL MODALITY

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ABSTRACT

Information and Communication Technologies (ICT) impact more and more all the aspects of personal, social and professional life of people, students and teachers included. All these changes influence teaching and learning processes too, they modify the relationships between teachers and pupils inside and outside the school and they generate new educational methodologies in order to improve mathematical skills. The purpose of this paper is to present an educational approach to L-system fractal theory (it is a no standard topic at the sixth form school level), to highlight the essential role that technology has had in learning and teaching it and to describe the project-based learning (PBL) methodology during the activity.

THE CONTEXT

L-system fractal is a geometrical topic managed by me in an extra-time course in the Liceo Scientifico “L. Siciliani” in Catanzaro (Italy) in cooperation with the University of Perugia, in a scientific project called “Mathematics and Reality” (M&R).

The national project M&R, managed by the prof. P. Brandi and A. Salvadori, proposes a new approach to mathematics education: the mathematical models and the real world.

The main actions in the project are mathematical laboratories, organized in each school, taking part in the projects, where the teacher proposes different themes, related to M&R, depending on the age of the pupils and the compulsory mathematical contents. For instance:

1. Linear mathematical models of reality (for 15-16 years old students);
2. Iterative functions and fractal geometry (for 16-17 years old students);
3. Nonlinear mathematical (exponential and trigonometric) models of reality (for 17-18 years old students).

THE EDUCATIONAL ACTIVITY: CONTENTS AND GOALS

My laboratory relates geometry, in particular fractal geometry, geometric properties of fractal figures and affinity transformations. The laboratory lasts 20 hours (two hours per week in a single lesson). Students voluntarily attend the course , they want to be involved in different educational methods about geometry. The activity is divided into two parts:

a) The presentation of Fractal Geometry Theory
   The following contents are presented by the teacher:
   - Iterative processes on geometrical figures; geometrical transformations and matrices;
   - Genetic code of a fractal; dimension of a fractal and properties;
   - L-system Fractals; turtle language.

b) The working-group presentations by the students
   During the course, the students are involved in:
   - Designing and creating multimedia presentations regarding the fractal theory contents;
   - Using free software and resources on the web, in order to learn and understand better the properties of fractals.

In the activity, the most important mathematical aims are:

For students

- Improving the study of geometrical transformations;
- Identifying the geometrical transformations and their codes;
- Making conjectures and individual simulations to create new fractals, in order to stimulate their creativity through the geometry;
- Using new technologies to improve a better comprehension of geometrical contents.

For teacher

- Developing a mathematics education opened to the external world, remaining anchored to the epistemology of the discipline;
- Giving students more autonomy in geometric knowledge;
- Using new technologies to better address the specific needs students may have;
- Involving students in a new educational approach to geometry

THE EDUCATIONAL ACTIVITY: METHODOLOGY

Project Based learning (PBL) is the principal methodology used during the second part of the activity.
This method has been practiced in order to facilitate self-learning through questions. The core concepts of PBL are: using professional knowledge, goal setting, problem resolution, and evaluation of the results. During an educational activity, PBL has the following properties:

• Starting learning with a problem or a question;
• Connection between the cognition and professional knowledge of the learners and the problems or questions;
• Learning in small groups;
• A self-oriented learning model;
• A different role of teacher during the activity: as helper, not as leader.
Students are encouraged in the activity to construct their own work by the geometric concepts learned in mathematics curriculum and in fractal geometry course.
However, although students have learned how to recognize a geometric transformation or to apply it to a geometric object in order to expand or scale down it, they may be unable to integrate the mathematical knowledge to design the final geometric work. Therefore, it is necessary to develop learning activities to guide students in creative design. (Wen-Haw Chen 2013)
During an activity, PBL methodology can be a good educational approach to achieve self-learning and independent knowledge in the students. PBL can be defined from different points of view. Barrows and Tamblyn (1980) define PBL as the process in which learners learn a about specific topics by understanding or solving specific problems. Other studies (e.g., Fogarty (1997)) have considered PBL as a course model that focuses on real-world problems. Schmidt (1993) and Walton and Matthews (1989) indicated that PBL is a learning method and can be used to explain the process of learning and teaching.

During PBL methodology, students and teacher take on the following roles:

- Students become stakeholders. This role contributes to the students' association to the old and new knowledge, understanding of the importance of special problem-solving strategies, and how to re-apply the problem-solving strategies in the future.
- The teacher is a trainer in cognition or meta-cognition. When teaching with a PBL approach, teachers must assume the role of a curriculum designer, learning or question-solving partner, supporter and director of learning, and evaluator of learning results.

PBL method derives from the hypothesis that learning is a process in which the learner actively constructs new knowledge along the basis of current knowledge. PBL provides students with the opportunity to gain theory and content knowledge and comprehension. PBL helps students develop advanced cognitive abilities such as creative thinking, problem solving and communication skills. These skills have been further improved through the use of technology, this one gives autonomy to the students that worked in this activity.
In our activity, we use the model consisting of four phases: Search, Solve, Create and Share (**Model SSCS**) (See Table 1)

Step	Content	Example in activity
Search	Brainstorming to identify problem, generate a list of ideas to explore, put into question format and focus on the investigation	Questions *a) How can affinity transformations convert in logical and formal rules of the L-system language?* *b) How does a fractal change, if the starting figure changes?* *c) How can we model our real world, using fractal patterns?*
Solve	Generate and implement plans for finding a solution, develop critical and creative thinking skills, form hypothesis, select the method for solving the problem, collect data and analyze.	Geometric concept: L-system fractal definition, rules and properties, turtle language.
Create	Students create a product in a small scale to the problem solution, reduce the data to simpler levels of explanation, and present the results as creatively as possible such using ppt presentation	Technology for: Researching information on the web, preparing presentation, creating images of fractal. In the activity, we use "Fractal Grower", it is a Java Software for Growing Lindenmayer Fractals. It was created by Joel Castellanos, Department of Computer Science, University of New Mexico http://www.cs.unm.edu/~joel/PaperFoldingFractal/paper.html
Share	Students communicate their findings, solution and conclusions with teacher and students, articulate their thinking, receive feedback and evaluate the solutions	National Congress of Mathematics for students called "Mathematics and Reality" at the University of Perugia in 2012 and 2014 Titles of presentations "*To make a tree… it takes an L-system fractal"* "*Fractal snowfall in Catanzaro*" Both works are an example of mixing affinity transformations and L-system fractals to model real world: in the first case the world of trees and in the second one the world of snowflakes.

(Tab.1)

THE ROLE OF TECHNOLOGY

The use of ICT enhances cognitive learning processes of students and allow themselves to discover in autonomy some geometrical properties. In fact, during the development of the L-fractal theory, we observe a dialectic between technology and the construction of the geometric content: there are some moments in which technology actually supports the comprehension of theory and there are others in which the use of technology transform theoretical properties in a repetitive and boring training of procedures.
Problem-Based Learning helps students develop creative thinking skills such as cooperative and interdisciplinary problem solving. Students learn to work both independently and collaboratively. During the activity, the teacher is a helper, not a leader, and the student has a high level of autonomy, because he can use many multimedia tools in order to create his presentation for the M&R congress. Every open question suggested to the student is the formulation of a single goal and the technology plays a relevant role, summarized in the table below (See Table 2), where goals, questions and content are connected and solved by a digital manipulating of geometric objects.

(Tab.2)

Questions	**Goals**	**L-system fractals content**	**Solution with technology**
How can affinity transformations convert in logical and formal rules of the L-system language?	Logical thinking	String, **ω** axiom, **p** production, turtle-language	Making graphical patterns
How does a fractal change, if the starting figure changes?	Problem solving ability	Affine transformation, contraction, rotation, translation	Recursive applications in geometric transformations
How can we model our real world, using fractal patterns?	Creative thinking	Logo-language	Creating patterns

MATHEMATICAL CONTENTS

A summary of the contents about fractal geometry and in particular L-system fractals are given below. Their theoretical discussion is simplified and adapted to the abilities, skills and age of pupils.

What a fractal is

Benoit Mandelbrot was the father of fractal theory. In the past, fractals were regarded as mathematical monsters, because of their unusual properties. In 1975, Mandelbrot called them fractals, from the Latin word *fractus*, meaning *fraction*. Fractal Geometry is around us, into the clouds, rivers, nature (Mandelbrot B. 1998).
Fractals are geometrical figures, characterized by unlimited repetition of the same shapes of a more lowered sequence. Fractal's proprieties are: self-similarity; scaling laws and not integer dimension There are different types of fractal: IFS (iterated function systems) and L-system fractals (Lindenmayer Fractals).

What an L-systems fractal is

L-systems fractals have been produced as a mathematical theory of growth plants After the incorporation of geometric features, plant models expressed using L-systems become detailed enough to allow the use of computer graphics for realistic visualization of plant structures and development processes. Aristid Lindenmayer (1925-1989) was a Hungarian biologist who developed a formal language called Lindenmayer Systems or L-systems to generate fractals. They were introduced as a theoretical framework for studying the development of a simple multicellular organisms, and subsequently applied to investigate higher plants and plant's organs.(Lindenmayer A., Prusinkiewicz P.1990). The central concept of L-systems is rewriting, it is a technique for defining complex objects by successively replacing parts of a simple initial object using a set of rewriting rules or productions. In 1968, Aristid Lindenmayer, introduced a new type of string-rewriting mechanism, subsequently termed L-systems.
In L-systems, we can define a string as an order triplet $\mathbf{G = (V, \omega, P)}$ in which:

1. $\mathbf{V}$ is a finite set of symbols called an alphabet;
2. $\boldsymbol{\omega} \in \mathbf{V^+}$ is a non-empty word called axiom ($\mathbf{V^+}$ is the set of all non-empty words over V);.
3. $\mathbf{P}$ is a finite set of production$\mathbf{: P \subset V \times V^*}$, $\mathbf{V^*}$ is the set of all words over V.

$\mathbf{P}$ defines how the variables can be replaced with combinations of constants and other variables. A production (a, $\boldsymbol{\omega}$) $\in$ P is written as a $\rightarrow \boldsymbol{\omega}$. The letter a and the word $\boldsymbol{\omega}$ are called the predecessor and the successor of this production, respectively. It is assumed that for any letter a ∈ V, there is at least one word **ω** ∈ V^* such that a → **ω**. If no production is explicitly specified for a given predecessor a∈ V , the identity production a → a is assumed to belong to the set of productions P.

Turtle language as graphical interpretation of L-system

One of the geometric systems that computer graphics used for the L-system's generation is called Turtle Geometry. The basic idea of turtle interpretation is given below.
A state of the turtle is defined as a triplet (x, y, α) where the Cartesian coordinates (x, y) represent the turtle's position, and the angle α, called the heading, is the direction in which the turtle is facing.
Given the step size *d* and the angle increment *δ,* the turtle can respond to commands represented by the following symbols:
1. **F** (it moves forward a step of length *d* the state changes to (x'= x+dcosα, y'= y+dsinα, α) A line segment between points (x, y) and (x', y') is drawn);
2. **f** (it moves forward a step of length *d* without drawing a line);
3. **+** (it turns left by angle δ the state changes to (x,y,α+δ));
4. **–** (it turn right by angle δ, the state changes to (x,y,α-δ)).
Given a string v, the initial state of the turtle (x_0, y_0, α_0) and fixed parameters *d* and *δ*, the *turtle interpretation* of v is the figure drawn by the turtle in response to the string v. Specifically, this method can be applied to interpret strings which are generated by L-systems. (Prusinkiewicz, P.1999).

L-SYSTEM FRACTALS: CODES AND IMAGES

Example 1

1. Axiom **ω: f-f-f-f** start angle 0°, turn angle 90° (it corresponds to the initiator or the starter figure of the fractal) (Fig.1a);
2. Production **p: f = ff-f-f-f-f-f+f** (it corresponds to the generator of the fractal) (Fig.1b);
3. Fractal at the fourth generation (Fig.1c)

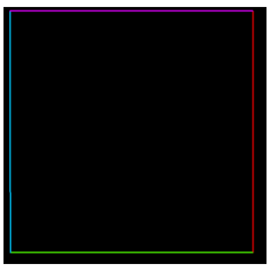


Fig.1a

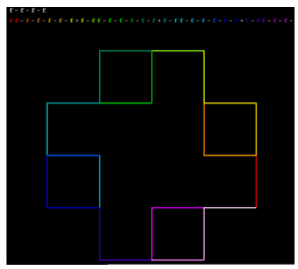
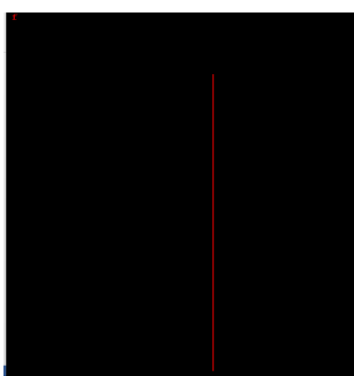


Fig.1b

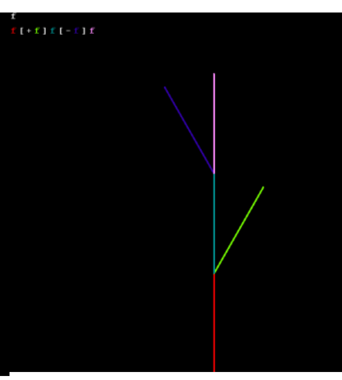
Fig.1c

Example 2

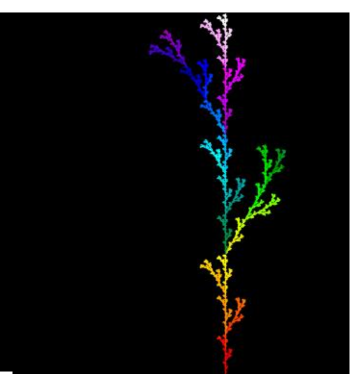
1. Axiom **ω: f** start angle 0°, turn angle 30° (it corresponds to the initiator or the starter figure of the fractal) (Fig.2a);
2. Production **p: f = f [+f] f [-f] f** (it corresponds to the generator of the fractal) (Fig.2b);
3. Fractal at the eighth generation (Fig.2c).



(Fig.2a)



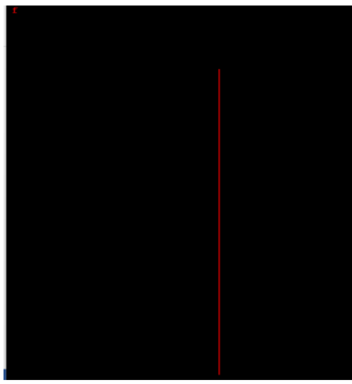
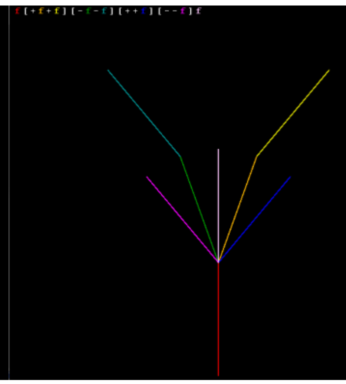
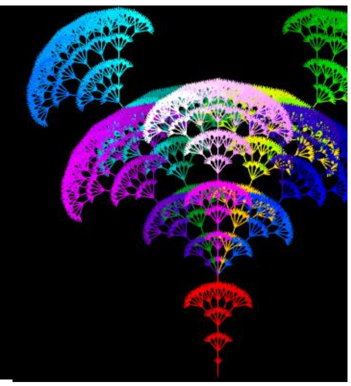
(Fig.2b)



(Fig.2c)

Example 3

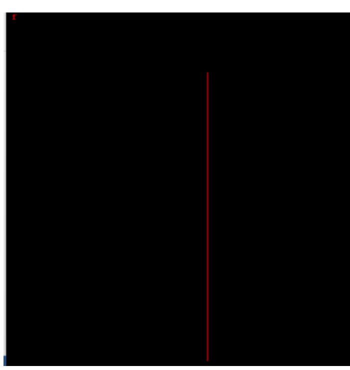
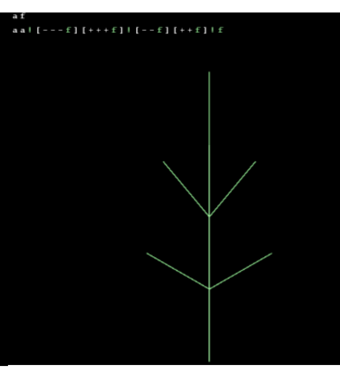
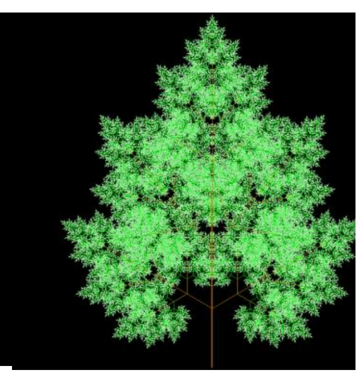
1. Axiom **ω: f[+f+f][-f-f][++f][--f]f** start angle 0°, turn angle 20° (it corresponds to the initiator or the starter figure of the fractal) (Fig.3a);
2. Production **p: ff[++f][+f][f][-f][--f]** (it corresponds to the generator of the fractal) (Fig.3b);
3. Fractal at the sixth generation (Fig.3c).

(Fig.3a) (Fig.3a) (Fig.3a)

Example 4

1. Axiom **ω: af** start angle 0°, turn angle 20° (it corresponds to the initiator or the starter figure of the fractal) (Fig.4a);
2. Production **p: a![---f][+++f]![--f][++f]!f** (it corresponds to the generator of the fractal) (Fig.4b);
3. Fractal at the seventh generation (Fig.4c).

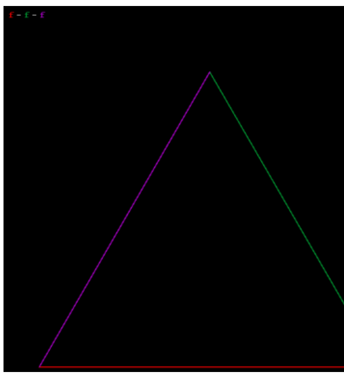




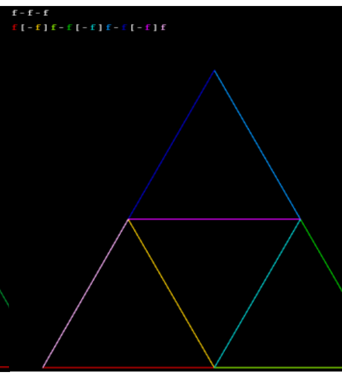
(Fig.4a) (Fig.4b) (Fig.4c)

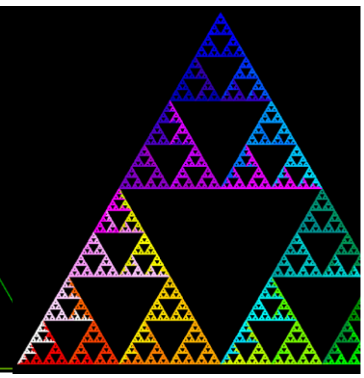
The examples below clarify the question: *How does a fractal change, if the starter figure changes?*

Example 5

1. Axiom **ω: f-f-f** start angle 90°, turn angle 120° (it corresponds to the initiator or the starter figure of the fractal) (Fig.5a);
2. Production **p: f[-f]f** (it corresponds to the generator of the fractal) (Fig.5b);
3. Fractal at the eighth generation (Fig.5c).

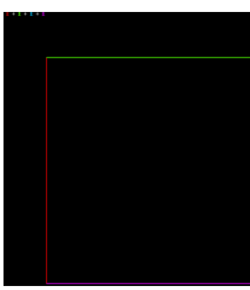

(Fig.5a)

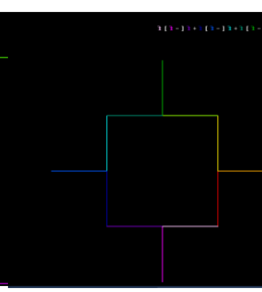

(Fig.5b)

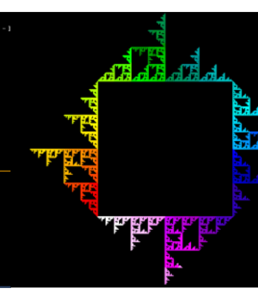

(Fig.5c)

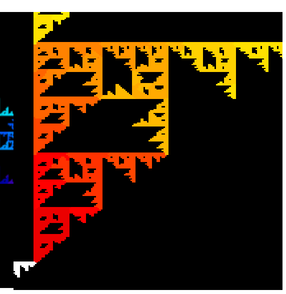
Example 6

1. Axiom **ω: f-f-f-f** start angle 90°, turn angle 120° (it corresponds to the initiator or the starter figure of the fractal) (Fig.6a);
2. Production **p: f[-f]f** (it corresponds to the generator of the fractal) (Fig.6b);
3. Fractal at the eighth generation (Fig.6c)
4. A detail of the fractal (Fig. 6d). Even if the starter figure changes, the fractal does not change.


(Fig.6a)


(Fig.6b)


(Fig.6c)


(Fig.6d)

CONCLUSION

We presented an approach to enhance the effectiveness of geometry teaching, in particular L-system fractal theory, through the PBL model incorporated creative design and the important role of technology in it. Our goal has been to present PBL as an instructional model that could encourage the creative design during the geometry teaching and the learning process. We hope it will improve geometry teaching and help students to integrate and apply the learned knowledge.

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