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Gregory Makrides, Elpida Christou

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TOWARDS A PERFECT VOTING SYSTEM

Alex Brekoulakis*, Elena Karachaliou, John Stratoulias*****
The Moraitis School, GR – 15452, Athens, Greece
*alexbrek13@yahoo.gr, **kar.elena99@gmail.com, ***ystratoulias@gmail.com
Supervisor: Dr Kostis Andriopoulos

ABSTRACT

Choosing a voting system may seem a simple-looking procedure, however it has to take into account many different social and political parameters as well as some mathematical contradictions. Thus, whether a voting system is considered fair or not is a debate that has been concerning researchers for decades. In this paper, we examine what makes a voting system fair, successful and effective. In order to achieve this, we present some of the most well-known voting systems and explain their mechanisms. Then we present some basic mathematical properties-criteria of social choice procedures. After all, the satisfaction or not of such properties is what determines the fairness of a voting system.

The question that arises is: "Can a voting system fulfill all these properties?" We show that all voting systems fail to satisfy some of the properties and we finally mention the impossibility theorems that prove that there is no social choice procedure that satisfies all properties, for example the Gibbard-Sattherthwaite theorem which can be practically applied. However, by examining the legitimacy of such claims, we discover voting paradoxes that arise in many hypothetical election scenarios, such as contradictions between some of the set criteria, and also scenarios in which it is impossible to fulfill some of these criteria, the existence of which seems to dictate that there is no perfect voting system. Or is there?

INTRODUCTION

Elections are possibly the most important part of democracy, since elections constitute the indirect way through which citizens govern a country, which makes an electoral/voting system also very important. Such a system is usually defined as the procedure through which votes are converted to seats in a country's parliament, a company's board, etc. However, in this paper we are going to examine voting systems in the sense of the procedure through which the votes are actually casted and valued, e.g. if the voter can choose from one or more candidates, if the voters grade the candidates with points etc.

Since a voter's preferences remain the same no matter which system is used, the choice of a voting system is perceived as a trivial matter that has practically no effect on the outcome. However, as we demonstrate below, this choice will not only largely affect the outcome, but it will also define whether the election outcome is fair, representative and, consequently, democratic. Also, a major factor is the satisfaction or non-satisfaction of a plethora of criteria that a voting system needs to fulfil. Some of the most commonly accepted criteria are also demonstrated below, in the second section of this paper.

As made apparent by the above statements, the making of a voting system is a complex procedure. It requires elaborate work, testing and reconsideration, so that the final product, by satisfying the soon-to-be-mentioned criteria, cannot be subject to manipulation, contradictions and misrepresentation in any scenario. However, in the third part we present observations and theorems which claim that no voting system can satisfy all, or certain sets of, criteria, dictating that absolute fairness and accuracy in representation can never be achieved. Moreover, sometimes even logical contradictions can occur, meaning that results can go against common logic at first glance. In this case, such a phenomenon is called a “paradox” and the examination of such phenomena comprises the final part of this paper.

VOTING SYSTEMS

In this section we mention some of the most well-known voting systems. These are the plurality, Condorcet's method, Borda count, first past the post system, single transferable vote and range voting.

Plurality:

Plurality is divided into two categories: relative and absolute majority. The relative majority declares as a winner the candidate with the largest number of votes. However, it is not necessary for the winner to have more than half of the votes. According to the absolute majority, the winner is obligated to have more than half of the votes otherwise a two-round system or a ranked voting system is applied. The two-round system, which is called plurality with runoff, is applied in many countries such as France, Portugal, Brazil and Cyprus [1].

4 voters	**3 voters**	**5 voters**	**4 voters**
A	B	D	D
B	C	A	B
C	A	B	C
D	D	C	A

Table 1: Voters' preferences

For example, if the voters' preferences are as shown in Table 1 it means that 4 voters (the first column) would prefer as a winner candidate A, their second preference would be candidate B, their third candidate C and last one candidate D. Three other voters (the second column) would vote for candidate B as the winner, C would be their second choice and so on.

According to plurality, taking under consideration the first preferences of the voters, the winner should be candidate **D** as he has the absolute majority of the votes (9 out of 16).

Condorcet's method:

During the 18th century Marquis de Condorcet, a French philosopher, mathematician and political scientist, developed another voting system which suggested pairwise comparisons between the alternatives. That is to say that by performing one-on-one contests between the alternatives we ultimately reach to the final winner which is the one who wins all the other alternatives on these one-to-one contests [3, 12].

For example, four candidates are ranked by 16 voters as shown in Table 1. First we compare A and B, then B and C and so on. When we compare candidates A and B we ignore the other candidates (C and D) and we have A with 4+5=9 votes and B with 3+4=7 votes. Thus candidate A wins candidate B. Following the same procedure for the rest of the candidates we get:

A>B (9>7)	A>C (9>7)	
B>C (16>0)	B<D (7<9)	} D>A>B>C
C<D (7<9)	A<D (7<9)	

According to Condorcet the winner should be candidate **D**.

Borda Count:

During the 18th century Jean-Charles de Borda a French mathematician, physicist, political scientist and sailor developed a new voting system [13]. According to this voting system each voter sets an order of preference for the alternatives. Then we assign points to every alternative depending on their position. For example for the 4th (last) position we assign 0 points, for the 3rd 1 point etc. [3, p.6-7]. The candidate with the most points wins. This system does not always satisfy the Condorcet's winner criterion, the Plurality criterion and the Independence of Irrelevant Alternatives (which are presented in what follows). On the other hand, it satisfies the Monotonicity and Condorcet's Loser criteria and takes regard to the comparative preferences of the voters towards the candidates. The Borda count is, surprisingly enough, only used for the election of two ethnic-minority members of the National Assembly of Slovenia, the Parliament of Nauru, the presidential elections in Kiribati and various private organizations and awards, such as the NBA's Most Valuable Player award and the Eurovision Song Contest [1].

If, for example, the preference lists for the 16 voters are as shown in Table 2, the results, according to the Borda count, are:

A → 4*3 + 3*1 + 5*2 + 4*0 = 25
B → 4*2 + 3*3 + 5*1 + 4*2 = 30
C → 4*1 + 3*2 + 5*0 + 4*1 = 14
D → 4*0 + 3*0 + 5*3 + 4*3 = 27

4 voters	3 voters	5 voters	4 voters
A [3]	B [3]	D [3]	D [3]
B [2]	C [2]	A [2]	B [2]
C [1]	A [1]	B [1]	C [1]
D [0]	D [0]	C [0]	A [0]

Table 2: Voters' preferences for Borda Count

The winner, according to Borda, should be candidate **B** with 30 points.

Coombs' Method:

Clyde Coombs, a psychologist in the 20th century, suggested another voting system where the voters rank all of the candidates on their ballot. If at any time one candidate is ranked first (among non-eliminated candidates) by an absolute majority of the voters, that candidate wins. Otherwise, the candidate ranked last (again among non-eliminated candidates) by the largest number of voters is eliminated. In other words, under instant-runoff voting, the candidate ranked first (among non-eliminated candidates) by the fewest voters is eliminated.

For example, as shown in Table 3, no candidate has the absolute majority (at least 9 votes) so we eliminate the candidate ranked last by most of the voters, that is candidate D. Now, those 5 voters, who had D as their first choice, vote for A. Candidate A has the absolute majority of the votes (9 out of 16) and gets elected.

First past the post system (FPTP):

In FPTP, voters cast a single preference vote, meaning they only vote for their top preference and the candidate that has a simple majority rather than an absolute majority is elected. This means that they do not need to have over 50%, instead just more than any other candidate. A downside of this voting system is that it essentially invites for strategic voting. This happens because supporters of minority parties or candidates, knowing there is little chance of their top preference to be elected, vote for a major party or candidate close to their preference so that the latter's percentage surpasses that of other major parties which are farther away from the voter's preference. This system is applied in Great Britain, USA, Canada and India [1].

Applying the FPTP system to Table 3 the results are:
A → 4 votes
B → 7 votes
C → 0 votes
D → 5 votes
Therefore, candidate **B** is the winner.

4 voters	3 voters	5 voters	4 voters
A	B	D	B
B	C	A	A
C	A	B	C
D	D	C	D

Table 3: Voters' preferences for Coombs

Single Transferable Vote (STV):

The STV is a more specific form of the FPTP and it is quite interesting as it elects more than one winner. For this system a minimum number of votes is set (the limit) and the voters are obliged to create their individual preference lists. Then, by adding the votes of each list for every voter we check whether any candidate has surpassed the limit or not. If he has then he is immediately a winner. Afterwards, the difference of his votes minus the limit is given to the second choice of the voters. If someone else surpasses the limit the same procedure is followed. If at some point there is no alternative that reaches the limit, the votes of the candidate with the less votes are shared at the second choice of these specific voters. These two procedures are repeated again and again as many times as needed [7]. The STV is applied in Ireland, Malta, Scotland, Australia and other government elections [1].

Range voting (also known as score voting):

In the RV system, voters grade all candidates using a numerical score (usually 1 to 10 or 1 to 100). The candidate with the highest average wins. This system is a single-winner voting system which means that only one candidate can be elected. Surely it reflects the preference of the voters better and the results are easy to understand and more accurate than other electoral systems. (A form of the system was used in some elections in ancient Sparta in which the candidate whose name was shouted louder was elected).

The ballot would look like this:

Governor Candidates	→	Score *each* candidate by bubbling a number (0 is worst; 9 is best)
1: Candidate A	→	⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨
2: Candidate B	→	⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨
3: Candidate C	→	⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

CRITERIA

When creating or seeking for a voting system, there is a plethora of criteria and properties desirable for a voting system to fulfil. However, as is further examined in the next section, it is probably impossible for a voting system to not violate at least one criterion or property. Because of this, we need to prioritize and rank criteria, and then choose our voting system based on our most highly preferred ones. There are three criteria ranked at the top by most sources, since they follow common sense, their fulfilment is rather easy to achieve, and they are inter-compatible. These are the Pareto condition, the transitivity and the regard to second choice. However, we mention in what follows some other criteria as well.

1. The Pareto condition:

A social choice procedure satisfies the Pareto condition when in case everyone prefers candidate B to C, then C is not considered the social choice or even is among the social choices in a tie, thus is not elected. For example, as shown in Table 4, C should never be elected since candidate B is preferred to C by all the voters [3, p.11-12].

4 voters	3 voters	5 voters	4 voters
A	B	D	D
B	C	A	B
C	A	B	C
D	D	C	A

Table 4: Voters' preferences; criteria

2. Transitivity:

The transitivity criterion dictates that if the majority of voters prefers A to B and B to C, then A is ranked higher than C. Mathematically speaking, if A>B and B>C, then A>C (that is similar to the transitivity property in Algebra).

3. Regard to second choice:

A voting system needs not depend only on the voters' first choice. In other words, there needs to exist a scenario and a set of votes in which the second choice of the voters influences the outcome. A characteristic example of a system that does not satisfy this criterion is Plurality.

4. The plurality criterion:

Plurality dictates that when a candidate has over 50% of the votes, then he/she should always be elected.

5. The Condorcet criterion:

A candidate in an election who would defeat every other candidate in a one-on-one or pairwise comparison is said to be a Condorcet winner. A voting system is said to satisfy the Condorcet criterion when it always elects the *Condorcet winner* (when there is one) [2].

6. Condorcet's loser:

Condorcet's loser is basically the opposite of Condorcet's winner. If A loses against all other candidates in a one-on-one comparison, then A is a *Condorcet's loser* and should not be elected.

7. Coombs' criterion:

According to Coombs, when a candidate is ranked last by the majority of voters, that candidate should not be elected. It is regarded as one of the voting systems that depict social choice the best.

8. Monotonicity:

A voting system is characterized as monotone or monotonic when in a scenario that A is the social choice, if someone changes his/her preference list in a way favorable to A then A should still remain the social choice. For example if in a set of votes that elects A, someone ranks B>A>C>D, and they change it to A>B>C>D, then A should still be elected, as depicted in Table 5 [3, p.12].

Voter 1	**Voter 2**	**Voter 3**
B	A	A
A	D	C
C	C	B
D	B	D

Voter 1	**Voter 2**	**Voter 3**
A	A	A
B	D	C
C	C	B
D	B	D

Table 5: voters' preferences; monotonicity

9. Independence of irrelevant alternatives:

The independence of irrelevant alternatives criterion dictates that when in a social choice procedure someone changes his preference list but he does not change the relative positions between two candidates then the relationship between them remains the same. For example, if someone ranks B>A>C>D, so that A>C, then, if they change their vote to D>A>C>B, it should still be A>C [3, p.12-13].

Voter 1	Voter 2	Voter 3
B	**A**	**A**
A	D	**C**
C	**C**	B
D	B	D

Voter 1	Voter 2	Voter 3
D	**A**	**A**
A	D	**C**
C	**C**	B
B	B	D

CONTRADICTIONS AND PARADOXES

According to Mark Sainsbury a definition for the word "paradox" is "an unacceptable conclusion derived by apparently acceptable premises" or, by the Oxford dictionary definition, "a seemingly absurd or contradictory statement or position which when investigated may prove to be well founded or true". The term seems to have derived from the Greek word "paradoxon" (Greek: *παράδοξον*) which means something contrary. The term probably appeared in the middle of the 16th century in Latin [5].

In this section we analyze and explain the contradiction between the Borda count, Condorcet's method and the plurality, the Condorcet's paradox and Arrow's theorem. For more information on paradoxes that arise in Voting Theory see [11].

BORDA COUNT CONTRADICTING CONDORCET AND PLURALITY

The Borda count system does not always satisfy the Plurality criterion and the Condorcet's winner criterion. Consider, for example, the voters' preference lists as shown in Table 6.

Number of voters and preferences		
14	**10**	**2**
A	C	D
B	B	B
C	D	C
D	A	A

Table 6: Borda's count, Condocet's method and plurality

Through the procedure of Plurality the winner is candidate **A** since he/she wins with 14 votes (absolute majority).

According to Condorcet's method the winner is again **A**, because he/she wins every other alternative on all one-on-one contests:

A>B (14>12)	A>D (14>12)	
B>C (16>10)	B>D (24>2)	**A**>B>C>D
A>C (14>12)	C>D (24>2)	

However, using the Borda count the points are:
A gets 14*3+10*0+2*0=42 points
B gets 14*2+10*2+2*2=52 points
C gets 14*1+10*3+2*1=46 points
D gets 14*0+10*1+2*3=16 points
B>C>A>D

We can clearly see that the winner is **B**.

For the same preference lists we applied three procedures and we got two different results. Therefore, the Borda count may violate the Plurality Criterion and the Condorcet's winner criterion [6].

CONDORCET'S PARADOX

voter 1	**voter 2**	**voter 3**
A	B	C
B	C	A
C	A	B

Applying Condorcet's method (comparing the alternatives) the results are:
A>B
B>C
C>A

Then it should be valid that A>B>C>A

This cyclic relationship between the alternatives prevents from concluding to a unique winner [4, p.15-22]. However, there are various methods to overcome such cycles (consult [11] and references therein).

ARROW'S THEOREM

Kenneth Arrow won a Nobel Prize in Economics [10] for, among other achievements, the following theorem that he stated and proved [14]:

A voting system is impossible to satisfy the following criteria at the same time:

1. There is no dictator (someone whose opinion overpowers all the other opinions)
2. If all the voters prefer A to B then at the final results A should be ranked higher than B (unanimity).
3. There is always only one winner/ there are no ties (universality).
4. It satisfies the Independence of Irrelevant Alternatives.

The problem is that Arrow defined a voting system as a system in which the voters classify the candidates (at least three distinct candidates). As a result Arrow's definition limits the range of voting systems in which it is applied and for example does not consider Range Voting as a voting system even though it is [8].

CONCLUSION

In this paper, we have presented some well-known voting systems, discussed the most important criteria that voting systems should (at least theoretically) fulfill and mentioned some contradictions and paradoxes.

As demonstrated above, Arrow's theorem is only applied for a class of voting systems as described in its definition. On the other hand, the Gibbard-Satterthwaite theorem states that, for three or more candidates and for every voting rule, either there is a single voter, who can determine the winner (dictator), or there is a candidate who can never win or the rule is open to (strategic) manipulations, in the sense that the sincere and strategic vote are not the same. That is to say that voting according to the real preference of the voter may not be the best strategic option. Hence, every voting system that selects a single winner either is manipulable or does not meet the assumptions of the theorem. The difference to Arrow's theorem is that it chooses a winner directly rather than a complete preference order of the candidates.

Therefore, it seems that there cannot be any absolute fairness in existing or future voting systems [9]. Or can there?

In recent years many scientists have constructed various electoral systems that try to equilibrate the strategic vote with the sincere vote and also satisfy several important criteria. However, since there is seemingly no formal referral of these systems in scientific journals and have not been applied anywhere, in our knowledge, their practical applications may only look distant. On the other hand there is the belief, or possibly the hope, that in the future there should appear a voting system (possibly a modification of existing ones) to make its real-life application possible, free of the limits set by the Gibbard-Satterthwaite theorem. Will we finally have the ultimate voting system? Only time can tell!

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VOTING PARADOXES

Dioni Kechribari*, Vasilis Kopanas, Christos Panagoulis***, Giannis Simantiris****, Sophia Spyropoulou*****, Konstantinos Tomaras********
The Moraitis School, GR-15452, Athens, Greece
*diony.kex@gmail.com ** kopanasvasilis@gmail.com ***chrispanagoulis@yahoo.gr ****
john99sim@gmail.com ***** sofia.spear16@hotmail.com ****** ctkonnos@gmail.com
Supervisor: Dr Kostis Andriopoulos

ABSTRACT

Many times we feel confident that our way of thinking is valid. What happens, however, when our seemingly unmistakable syllogism is proved wrong? In that case, the fundamentals on which we are based collapse and we end up reaching false conclusions. The situation gets worse when these false conclusions are used during electoral processes in order to make collective decisions. These "glitches" in electoral systems, which make it difficult for us to interpret the election result, are called voting paradoxes. In this paper we analyze the most well-known cases of electoral systems being self-cancelling because of their own paradoxical nature.

Initially, we analyze Condorcet's Paradox: although the majority of the voters would prefer candidate A to candidate B and candidate B to candidate C, the latter is still preferred over A. Afterwards, we present the No Show Paradox: it may occur that if some voters decide not to take part in the elections, the elected candidate will be closer to their preferences than in the case of them actually voting! Finally, we expand on the Dominated Winner Paradox: a candidate gets elected, despite the fact that each and every voter prefers another candidate to him.

The frequency at which all these paradoxes appear depends on a large number of factors. And, of course, when they do appear, they raise many questions on whether our society is as democratic as we think.

INTRODUCTION

A voting system is the method by which voters choose one of the available options and enforces rules on how these votes should be aggregated to yield a final result [1]. There are different kinds of voting systems which can be divided in two categories. In the voting systems of the first category, voters directly select the candidate of their choice. So, for example, in the majority system the winner is simply the candidate who has collected the highest number of votes. The second category consists of electoral systems in which voters are allowed to rank the available choices depending on their preference [2, p. 4]. In these systems, tables such as the following are used to present the data:

26 voters	6 voters	19 voters
C	B	A
A	C	C
B	A	B

Table 1: Voters' preference over three candidates

In Table 1 we see that 26 voters rank C at the first position, A at the second position and B at the last one. Similarly, we can derive information from the other two columns.

Sometimes, however, the results of the voting systems come in contrast with our common sense. To be more specific, sometimes in the electoral process some unpleasant surprises appear that are called voting paradoxes [2, Preface].

Paradoxes lead us to unexpected conclusions that seem to be in contrast to the views expressed by the voters. So, due to that, various questions arise concerning the reliability of the applied electoral systems. In this paper we analyze some of the most well-known paradoxes, namely Condorcet's paradox, the No show paradox, and the Dominated winner paradox. Afterwards we use Condorcet's voting system to present some of the factors that affect the frequency at which paradoxes appear. Finally, we mention a characteristic case of a paradox that has appeared in real elections and, based on that, we present our conclusions.

CONDORCET'S PARADOX

Marquis de Condorcet was a famous French mathematician and philosopher of the 18th century. Having been influenced greatly by the Enlightenment and the French Revolution, Condorcet supported the opinion that all people have the same rights regardless of their sex and origins [4]. However, he became well-known for the introduction of his own electoral system (Condorcet's system) and his research on the paradox which is analyzed below. But first, we examine Condorcet's electoral system itself [8].

Condorcet's system is based on the idea that the winner of the elections should be the candidate who is preferred over each of the other candidates by the majority of the voters. For that reason we compare all candidates in pairs, two at a time, so as to find who the winner is in each one-to-one battle. The candidate who beats all of his opponents through this process is called a Condorcet winner.

Suppose that we have the results of an election among the candidates A, B, C presented in Table 2:

40 voters	20 voters	18 voters	22 voters
A	B	B	C
B	A	C	B
C	C	A	A

Table 2: Condorcet's system

We see that 20+18+22=60 voters prefer B to A (B>A), while 40+20+18=78 voters prefer B to C (B>C). So, since B outstands both of his opponents, according to Condorcet he should be the winner.

In this particular example, Condorcet's system is applied without problems and the winner is undoubtedly the right one. However, as we can see in Table 3, this may not always be the case:

30 voters	5 voters	25 voters	22 voters	18 voters
A	B	B	C	C
B	A	C	A	B
C	C	A	B	A

Table 3: Condorcet's paradox

In this instance, we find that 52 voters prefer A to B (A>B), and 60 voters prefer B to C (B>C). From the above data, we could infer using the transitive property that, since the majority prefers A to B and B to C, the winner should be A. However from the last 3 columns we can see that 65 out of 100 voters prefer C over A!

The problem is that in Condorcet's electoral system, we cannot use the transitive property, according to which, if A>B and B>C, then A>C. This is because B does not have the same value in both inequalities (in the first one it takes the value of 48 and in the second the value of 60). As a result, in order to find a winner in Condorcet's system, the existence of a candidate that beats every opponent in a one-to-one comparison is necessary.

In case such a candidate does not exist, a cycle is created among A, B and C, as we can see below:

A> B> C>A

In this scenario we cannot elect any candidate since each one of them is defeated in an encounter by one of his opponents [2, page 15-16].

However, Condorcet did suggest a method to overcome this problem [2, page 16]. According to this method, we may ignore the preference relation that is supported by the smallest number of voters. That would leave us with one less inequality and we would be able to elect a candidate.
In our example:

- 52 voters prefer A to B (A>B)
- 60 voters prefer B to C (B>C)
- 65 voters prefer C to A (C>A)

Since the weakest inequality is the first one, we ignore it and we are left with the final two. Therefore, we can now conclude that since B>C and C>A, the inequality that reflects the voters' preference over the three candidates in the most accurate way is the inequality B>C>A. In this case B is the most appropriate candidate to be elected. Although this method succeeds in overcoming Condorcet's paradox in many cases, we are still unable to apply it when some of the inequalities are of equal size.

If there is a larger number of candidates, we may keep deleting the weakest inequalities (one at a time) until we are left with a number of inequalities that allow us to find a winner. This is called the successive reversal procedure. It should be mentioned however, that this method has been criticized by some authors, which has led to a considerable number of alternative solutions [2, p. 16-18]. For example, Saari [10] proposes the following procedure: Condorcet cycles cancel out and should be deleted. Therefore, 22 voters in columns 1, 3 and 4 should be removed from any analysis since their preferences cancel out completely. This consideration promotes candidate C and according to Saari he/she should be elected.

NO SHOW PARADOX

Consider the two-round voting system. In this system, the voters are asked to choose between some candidates, for example A, B and C. If one of the candidates is the first choice for more than half of the voters, he wins the elections. In case there is no such candidate, the process is repeated between the two candidates who collected the highest number of votes, in order to find the final winner. Suppose the following data table:

26 voters	49 voters	25 voters
A	B	C
B	C	A
C	A	B

Table 4: Two-round system

In this example (see Table 4) it is obvious that none of the three candidates is the first choice for more than half of the voters (at least 51 out of 100). Therefore, the process will be repeated between the candidates A and B, who are placed at the top of the preference list for the largest number of voters (26 and 49 respectively). But then, we see that although in the first round B beats A with 49 against 26 first votes, in the second round A wins with 25+26=51 votes.

This happens because C has been eliminated and those 25 voters who placed him at the top of their preference list in the first round now vote in favor of A (their second choice).

Suppose now that 47 of the 49 voters of the second column decide not to participate in the elections. Then, the data table is as follows [2, p 49]:

26 voters	2 voters	25 voters
A	B	C
B	C	A
C	A	B

Table 5: No show paradox

In this case, we notice that the elections will be repeated between A and C, since none of the candidates is the first choice for 27 or more voters. So, in the second round between A and C, the latter wins with 25+2=27 votes.

By taking a closer look at the two tables we see that, if those 47 voters did participate in the elections, the winner would be A, who is found at the bottom of their preference list. However, if they did not participate in the elections the winner would be C whom they prefer to A. Therefore those voters may improve the result of the elections for themselves by actually avoiding taking part in the voting procedure! This is the so-called "No show paradox" [2, 11].

DOMINATED WINNER PARADOX

To illustrate the Dominated winner paradox we make use of the successive elimination voting system. Suppose that three voters (V_1, V_2 and V_3) wish to elect one of the candidates A, B, C and D. Initially, each of the voters writes down the names of the four candidates in a list, according to his preferences. Afterwards, two of the four candidates are compared in order to find which of them is ranked higher than the other by the majority of the voters and the winner moves forward to the next pairwise comparison between himself and a third candidate. Then, we once again find which of the two is placed at a higher position by most of the voters and that candidate is compared with the fourth one in the final round, so as to find which of the two will be ultimately elected.

At first, the aforementioned method seems fair due to the fact that each of the pairwise comparisons serves as a means to eliminate a candidate that is ranked lower than another and, therefore, should not be elected. One could think that this procedure is supposed to lead us to the final candidate who is not ranked below anyone else. However, this may not always be the case. We examine the following table [3, p. 108]:

V_1	V_2	V_3
B	C	A
A	B	D
D	A	C
C	D	B

Table 6: Dominated winner paradox

In the first pairwise comparison between A and B we see that two of the three voters (V_1 and V_2) rank B higher than A, therefore A should not be considered a viable option. Next, in the comparison between B and, say, C, the latter wins since two of the three voters (V_2 and V_3) place him above B, so he should move forward to the final round. Finally, between C and D it is clear that D should be the winner since he is preferred by the majority of the voters (V_1 and V_3). It is therefore obvious that in this case, using this specific voting procedure, D is the one who gets elected.

Although we followed the procedure step-by-step as it was described above, by taking a closer look at Table 6 we see that the result of the elections cannot be considered a fair one. To be more specific, in a pairwise comparison between A and D it is clear that each of the voters ranks A higher than D, yet D was the one to be selected as the winner. In other words, while

A was placed at a higher position than D by each of the voters, he lost to him in the elections. This is called a violation of the Pareto criterion. According to the Pareto criterion, "if all voters strictly prefer x to y, then y should not be elected" [2, p. 88]. In this case, since all voters prefer A over D, D should not be elected and doing so would be considered unfair.

Just as Condorcet's paradox, the dominated winner paradox is formed due to the intransitivity of the preference relation between the pairwise preferences of the voters. That is to say that the pairwise inequalities $A<B$, $B<C$ and $C<D$ do not necessarily mean that $A<D$ since the number of votes in favor of each of the four candidates is different in each of the inequalities. Another factor that affects the result of the voting process is the order in which we compare the candidates. If, for example, the initial order of candidates were B-C-D-A (rather than A-B-C-D), then A would be the winner instead of D. Therefore, comparing the candidates in a different order may lead us to different results, which is the main reason why this is not considered to be a very reliable voting system.

FREQUENCY OF PARADOX APPEARANCE

The paradoxes that have been analyzed above clearly do not appear in every situation. It is therefore reasonable to wonder in what frequency these paradoxes appear and how often electoral systems work without facing such problems? It is not possible to give an accurate answer to this question since the factors that affect it are multiple and cannot always be defined. The voters are humans and their will is influenced by various personal factors that affect their choices, and hence, the probability of voting in one specific way is neither stable nor predictable. But even if everyone voted randomly (and therefore the probability for each case to appear were stable), the answer to this question would still be influenced by factors such as the number of the voters, the number of the candidates and the electoral system itself. However, for some specific values of the parameters mentioned above (number of voters and number of candidates) we can calculate the probability of the appearance of Condorcet’s paradox. The results are shown in Table 7 [2, p. 25-27]:

	3 candidates	**5 candidates**	**13 candidates**	**25 candidates**
3 voters	6,7%	16,0%	38,5%	52,5%
7 voters	7,5%	21,5%	50,0%	65,5%
17 voters	8,3%	23,7%	54,1%	70,0%
29 voters	8,5%	24,3%	55,3%	71,2%
39 voters	8,6%	24,6%	55,7%	71,7%

Table 7: Probability of appearance of Condorcet's Paradox

This probability, as we can see in Table 7, increases depending on the number of voters and the number of candidates taking part in the elections. This means that if there is a specific number of voters (say 3), as the number of candidates increases from 3 to 25, so does the probability of appearance of Condorcet's Paradox (rising from 6,7% to 52,5%). Similarly, for a specific number of candidates (say 3) as the number of voters increases from 3 to 39, so does the probability (starting from 6,7% and gradually reaching 8,6%). An important thing to mention here is the fact that this probability increases in a much quicker rate in the case in which only the number of candidates increases than in the case in which only the number of voters increases. This means that Condorcet's system is significantly less applicable in elections with a large number of candidates. It should be mentioned however, that in different voting systems, different criteria affect the chance for a specific paradox to appear, so the above conclusions should not and cannot be generalized.

BORDA'S ARGUMENT

Borda, a French mathematician who lived at the same time period as Condorcet, noticed that the winner of some electoral systems might be the loser in Condorcet's system. Such an example is presented in Table 8 [2, page 12]:

1 voter	7 voters	7 voters	6 voters
A	A	B	C
B	C	C	B
C	B	A	A

Table 8: Borda's system

According to the relative majority system, one of the most commonly-used voting systems, the winner should be the candidate who is the first choice for the highest number of voters. That being said, since A gets 1+7=8 votes, B gets 7 votes and C gets 6 votes, A should be the winner. However, by using Condorcet's system we see in the last two columns that 7+6=13 out of 21 voters rank candidates B and C over A. In other words, the elected candidate of a very popular voting system (relative majority) may be one that loses in a one-to-one comparison with each of his opponents. For more information on voting systems check [12].

In order to deal with that problem, Borda introduced a new electoral system [9]. In that system every candidate gets a specific number of points depending on his position on the preference list of each voter. The winner is the candidate who gathers the most points. Therefore, if we assign 2 points to the candidates at the first place, 1 point to the candidates at the second place and 0 points to the candidates at the third place, then:

A gets 1x2+7x2+7x0+6x0 = 16 points
B gets 1x1+7x0+7x2+6x1 = 21 points
C gets 1x0+7x1+7x1+6x2 = 26 points

It is clear that C should be elected.

Although Borda manages to eliminate the possibility that the elected candidate is a Condorcet loser, in some cases the Borda-winner and the Condorcet-winner are not the same. In other words, by using Borda's system we will never elect a candidate who loses in every pairwise comparison against his opponents. At the same time though, this does not guarantee that Borda's and Condorcet's systems will always elect the same candidate. This leaves a lot of room for debate on which of the two voting systems is the most fair [5].

THE 2000 U.S. PRESIDENTIAL ELECTIONS

Finding the differences between voting systems and trying to discover how paradoxes are formed are two research areas with many important applications in real life. That is because voting paradoxes do not exist only in theory, but they can often affect the voting results of real elections. One of the most well-known cases in which such a thing happened was the U.S. presidential elections in 2000 [7].

The contest was between Republican candidate George W. Bush, the governor of Texas, and Democratic candidate Al Gore, the Vice President. Even though Al Gore received 543,895 more popular votes than George Bush, he came in second in the electoral vote! This marked the fourth election in U.S. history in which the eventual winner failed to win a plurality of the popular vote.

The voting system applied in the United States, plurality of the popular vote, is not as simple a system as most people think. In this system, each voter selects his favorite candidate.

Afterwards, the candidate who collects the highest number of votes in each of the 50 states is considered the winner of that state and he wins all electoral votes of that state. Each state has a specific number of electoral votes depending on its population and the total number of these votes is 538. The candidate who gets more than 50% of the electoral votes (270 or more) is the one who becomes the president [6].

As the final national results were announced the morning after the elections of 2000, it became clear that Bush had won a total of 246 electoral votes, while Gore had won 255 votes. 270 votes were needed to win and the voting results of only three states remained to be announced. The two smaller states - New Mexico (5 electoral votes) and Oregon (7 electoral votes) - could not possibly affect the final outcome of the elections since each candidate needed more than 12 votes to win. Florida's 25 electoral votes, however, were all that each candidate needed to reach 270 electoral votes and become the president. Although Gore was the winner in both New Mexico and Oregon, Florida's statewide vote could reverse the situation. At the end Bush won Florida state where the margin of his victory was less than 0,01% (0.0092%) and he became the President [7].

CONCLUSIONS

It can be easily understood, from all mentioned above, that the election of a candidate or the process of making a decision using democratic means is anything but easy. There are various parameters and factors that, depending on the weight we place on them, may lead us to a different election result. This fact begs for several questions, which makes us reconsider several of our intuitive thoughts.

To begin with, to which extent is an electoral system evenhanded? Is it possible that the election results are affected by external factors and as a result we sometimes come to terms with decisions that, although they seem to have been taken democratically, express only a minority? And in such a case, are there certain people who would be able to influence the election result and change it to their advantage (strategic manipulation)?

Are actually politicians, businessmen, the media and all the other authority factors as uninvolved in electoral procedures as we think, or do they choose each time a different voting system, so as to have the election result interpreted and assessed in a way that enables them to maintain and maximize their social and financial power? And if so, would it be possible for anyone to realize that the election result is being altered, given the fact that such phenomena have been considered as paradoxes by even some of the greatest mathematicians-scientists of all times? All these questions lead us to think over the meaning of democracy nowadays.

Is democracy a political system as effective as we claim? Is it able to hold its ground and protect the decision making procedure so as to satisfy the public desire to the biggest extent possible? Or is it another failure of human civilization?

In conclusion, will the human race ever be able to overcome these paradoxes and create a really fair and by all mathematical criteria imposed electoral system? Or is it inevitable that such uncertainties will never be dealt with, which would serve as a reminder of the fact that we cannot, whatsoever, reach perfection? Maybe these paradoxes are evidence of the fact that human civilization is always under formation and it will never cover efficiently all of our needs.

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CRIME THROUGH REGRESSION MODELS

Eleni Nicodemou*, Maria Chrysanthou*, Lucy Shahabian*, Christos Papaioannou*, Alexandros Kokkinolambos*, Theodora Karaoli*
The Grammar School, Nicosia (michalis.gavrielides@thegrammarschool.net)
Teachers: Michalis Gavrielides, Noni Hadjicharou

ABSTRACT

Regression is a method of measuring the relationship between the mean value of one variable and the corresponding values of other variables. In cases where more than one factor affects a specific variable, multiple linear regression is used. This gives us the opportunity to study the relationship of more than one explanatory variable with one response.
The data set analysed in this project concerns the crime rates (response variable) of 28 countries of the European Union during the year 2011. The factors (explanatory variables) that may affect the crime rates which will initially be regressed on the response variable are Unemployment, Population, Government expenditure on public order and safety, Degree of Urbanization and Job Vacancy. However, not all of the above explanatory variables may affect the crime rates therefore through the selection procedures, we will investigate the 'offer' of each explanatory variable to the response variable.
Furthermore, when analysing a linear model through the regression analysis there are some assumptions that must be taken into consideration such as the linearity between response and explanatory variables, constant variance of the residuals of the model, independence between the errors of the model and the normality of the errors.
Throughout the years, crime has been a part of our everyday lives. Who would have thought that regression could be a method to prevent it?

1. THE MULTIPLE LINEAR REGRESSION MODEL

The multiple linear regression is used in the case where it is necessary to define the relationship between one response and more than one explanatory variables. The matter is how well the explanatory variables can interpret the variability of the response variable. To be more specific, below we present an example of a linear model that can be interpreted by the multiple regression analysis.

$$\boldsymbol{y = a + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \varepsilon}$$

where:
y : The response variable of the model
α : The intercept of the model
$\beta_1 , \beta_2 , \dots , \beta_n$: The coefficients of the model
$x , x_2 , \dots , x_n$: The explanatory variables of the model
ε : The random error term of the model

2. THE VARIABLES OF THE DATA SET

The data set of the following analysis[1] refers to the crime rates of 28 countries of the European Union measured at the end of year 2011. Therefore, the variables included in the following analysis consist of 28 observations each and all of them are measured in the same period of time. We totally use 6 variables for our model out of which 1 is the response variable and the remaining 5 are the explanatory variables.

The response variable of the following analysis is crime, measuring the total number of crimes recorded in the European Union by the police in 2011. The first explanatory variable is the unemployment rate of European Union and is expressed as a percentage. The unemployment rate is the number of unemployed workers divided by the total civilian labor force. A person is considered to be unemployed when they are able and willing to work but yet unable to find a job. The second explanatory variable is the population of each one of the 28 countries. The third explanatory variable is an index called expenditure on public order and safety and includes the expenses of each country on safety as a percentage of the total governments' expenses. The fourth explanatory variable is again an index called Degree of Urbanization, expressing as a percentage the number of people leaving in high dense areas. The fifth and last explanatory variable is the Job Vacancy which measures the number of unoccupied positions either in public or private sector.

In order to be more flexible in presenting the results of the following analysis, we summarise in table 1 below, the six variables together with their symbols and description.

TYPE	SYMBOL	DESCRIPTION
response	y	Crime
explanatory	x_1	Unemployment
explanatory	x_2	Population
explanatory	x_3	Expenditure
explanatory	x_4	Urbanisation
explanatory	x_5	Vacancy

Table 1: The type, symbol and description for the six variables of the data set

3. ANALYSIS OF THE DATA SET

3.1. CORRELATIONS AND COLLINEARITIES

The first step in our analysis is the presentation of the correlation matrix which consists of the correlation coefficients for all the pairs of the variables. In this matrix we include the response (crime) variable too in order to observe the linear relationship between the response and each one of the explanatory variables. Table 2 below presents the correlations between all the pairs of the variables.

	Crime	**Unemployment**	**Population**	**Expenditure**	**Urb/tion**
Unemployment	-0.183				
Population	**0.947**	-0.078			
Expenditure	-0.115	0.428	-0.021		
Urbanisation	0.328	-0.102	0.232	0.032	
Vacancy	0.334	**-0.479**	0.139	-0.320	**0.540**

Table 2: Pearson's correlation coefficients for all the pairs of the variables

The first thing to be observed in table 2 is that some correlation coefficients at the first column (headed by Crime) are positive and some are negative, therefore, we are informed that the response and each one of the explanatory variables are either positively or negatively correlated. As we can see the population variable have a strong linear correlation with the response variable as the correlation coefficient is estimated near to 1. Turning our concern to the correlation between the explanatory variables we can observe that the only correlation coefficient that may be statistically significant (close to 1) is the one between the Urbanisation variable and the Vacancy variable. Since these two variables are both explanatory we have the first indication about collinearity. Collinearity is a term that is used when two explanatory variables are highly associated and if we want to speak strictly in terms of statistics we have to say that both explanatory variables interpret the same part of the variability of the response variable. We can observe collinearity from the Pearson's correlation coefficient between the two explanatory variables; if the correlation coefficient is approaching 1 or -1 this is an indication of collinearity. The meaning of this large correlation coefficient is that the two variables are nearly identical; therefore they are not both informative for the response variable. Hence, it is obvious that we may not be able to use both Urbanisation variable and vacancy variable in the model. For getting an idea for the regression model that these data fits we present and discuss the **full** linear regression model (including all explanatory variables) in section 3.2 below.

3.2. REGRESSION OF THE FULL MODEL

For getting an idea for the regression model that these data fit we present the regression table for the full model that consists of all the variables in table 3 below. Therefore, the full model is of the following form:

$$\boldsymbol{y = a + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \varepsilon}$$

where notations for the response and the explanatory variables are available in table 1. Table 3 below presents the regression of the $\boldsymbol{y}$ (crime) variable on the five explanatory variables.

Model	Coefficients	t	p-value	VIF
(Constant)	-388615.663	-0.871	0.393	
Unemployment	-2671.396	-0.121	0.905	1.467
Population	0.059	17.028	0.000	1.060
Expenditure	-43302.358	-0.592	0.560	1.284
Urbanisation	2695.539	0.271	0.789	1.603
Vacancy	104747.552	2.458	0.022	2.018
$\boldsymbol{R^2 = 0.940}$				

Table 3: The full model together with some statistical tests

The second column of table 3 consists of the coefficients (parameters of the model) which form the equation of the full model as shown below.

$$\boldsymbol{y = -388616 - 2671x_1 + 0.059x_2 - 43302x_3 + 2696x_4 + 104748x_5}$$

The next two columns contain information on the hypotheses (null) that each parameter is equal to zero with the alternative one represent the fact that the parameter is different from zero. It is worthwhile to have a better look at the p-values column. As we can observe, according to the p-values (we use significant level of $a = 0.05$) for 3 out of the 5 testing hypotheses we do not have enough evidence to reject the null hypothesis (p-values are greater than 0.05). The exact meaning of the previous result is that some of the explanatory variables are redundant for the model since their parameters are equal to zero. But if we want to present a proper analysis we must be more careful since we have already seen an indication of collinearity and maybe all these tests about the parameters are not so reliable. The reason we assume that these tests may not be so reliable is that collinearities most of the times affect the signs of the estimations of the parameters as well as the variances of these estimations. Therefore, we first have to check for collinearity between the variables Urbanisation and Vacancy and then make the hypothesis testing about the parameters. Only then do we get reliable T-statistics about the parameters. The fifth column of table 3 presents the Variance Inflation Factors (VIF). As a rule of thumb if a Variance Inflation Factor is greater than 10 then the specific variable is involve in collinearity. As we can observe from the table 3 the VIF of Urbanisation and the VIF of Vacancy are both less than 10, therefore these two variables are not involved in collinearities. The last result to be noted form table 3 is the statistic R^2 which is actually a measure of goodness of fit of the model. The value of the R^2 varies from 0 to 1 with 0 indicating a very poor fitting of the model and 1 indicating the **best** fit of the model. As we can see from table 3 that value is 0.940 supporting a very good fit of the model. Although the R^2 statistic indicates a very good fit of the full model, according what we have seen from the T-statistics above, many of the variables are considered to be redundant. In order to choose which variables are statistically significant for the model we can make use of the variable selection procedures. Concluding this section we note that we do not proceed to check the residuals of the full model because the results of the parameters of the model would definitely change if any of the explanatory variables is removed from the model through the selection procedures.

3.3. VARIABLE SELECTION PROCEDURES AND REDUCED MODEL

Multiple linear regression models always include more than one explanatory variable. The first and most reasonable question that one may ask is how many explanatory variables should be used in a given regression model in order to get reliable results and a model that fits the data well. Generally speaking, there is not just one answer to this question and different analysts can conclude to different explanatory variables to be included in the model. However, there are some variable selection methods in regression literature which are widely used in the analysis of the data using regression models and we refer to them below.

3.3.1. BACKWARD ELIMINATION METHOD

The first method we are going to refer to is backward elimination. As the name suggests this method begins with a model that includes all candidate explanatory variables. Then the p-value is computed for the T-statistic for the coefficient of each variable as if it were the last variable to enter the model. The largest of these p-values is compared with a preselected value, a_{OUT} , and if the largest p-value exceed the a_{OUT}, that variable is removed from the model. This method terminates when the largest p-value does not exceed the preselected cutoff a_{OUT} value. We also have to note that we suggest $a_{OUT} = 0.1$ for the backward elimination method. The application of this method to our data set is available in table 4 below.

Model		Coefficients	t	p-value
		B		
1	(Constant)	-388615,663	-,871	,393
	UNEMPLOYMENT	-2671,396	-,121	,905
	POPULATION	,059	17,028	,000
	EXPENDITURE	-43302,358	-,592	,560
	URBANIZATION	2695,539	,271	,789
	VACANCY	104747,552	2,458	,022
2	(Constant)	-414770,570	-1,086	,289
	POPULATION	,059	17,433	,000
	EXPENDITURE	-45654,235	-,662	,515
	URBANIZATION	2513,202	,261	,796
	VACANCY	106850,362	2,807	,010
3	(Constant)	-446030,095	-1,254	,222
	POPULATION	,059	18,156	,000
	EXPENDITURE	-41035,030	-,628	,536
	VACANCY	112514,906	3,669	,001
4	(Constant)	-645257,976	-4,072	,000
	POPULATION	,059	18,370	,000
	VACANCY	118672,820	4,134	,000
$\boldsymbol{R^2(for\ model\ 4) = 0.939}$				

Table 4: The backward elimination method

From table 4 we can see that the backward elimination method terminates after four steps. In the second step, the method excludes the Unemployment variable because, as we can see, this variable has the largest p-value of all the other variables in the first step of the method. In the third step the variable Urbanization is removed from the model because it has the largest p-value of all the other variables in the second step. The backward elimination algorithm terminates in the fourth step by excluding 3 explanatory variables overall. The model that this method suggests is as follows:

$$\boldsymbol{y = -645258 + 0.059x_2 + 118673x_5}$$

A result that is of primary importance is the value of R^2 which takes the value of 0.939. The interpretation of this result shows the necessity of reducing the explanatory variables of the model as we explain the same part of the variability of the response variable with only two explanatory variables instead of five.

3.3.2. FORWARD SELECTION METHOD

This method begins with the assumption that there are no variables in the model other than the intercept. An effort is made to find an optimal subset by inserting variables into the model one at a time. The first variable selected for entry into the equation is the one that has the largest correlation with the response variable. This is also the variable that will produce the smallest p-value for the testing of the coefficients of the explanatory variables. This variable is entered if the p-value does not exceed the preselected value a_{IN}. The second variable chosen for entry is the one that has the largest correlation with the response variable after adjusting for the effect of the first variable entered on the response variable. We note that we suggest $a_{IN} = 0.25$ for the forward selection method. The application of this method to our data set is available in table 5 below.

Model		Coefficients	t	p-value
		B		
1	(Constant)	-114931,822	-,971	,341
	POPULATION	,061	15,033	,000
2	(Constant)	-645257,976	-4,072	,000
	POPULATION	,059	18,370	,000
	VACANCY	118672,820	4,134	,000
	$R^2(for\ model\ 2) = 0.939$			

Table 5: The forward selection method

According to the forward selection method the first variable to be entered in the model is Population, since this is the variable with the smallest p-value (given that this p-value is smaller than the a_{IN}). The second and final variable to be entered in the model is the Vacancy variable. The two explanatory variables that have been selected by the forward selection method are exactly the same variables that have been selected by the backward elimination method. Therefore, since both methods conclude to the same reduced model, we present the final model below and we enter the explanatory variables at the exact same order that they have been selected by both methods.

$$y = -645258 + 0.059x_2 + 118673x_5$$

3.4. GOODNESS OF FIT OF THE REDUCED MODEL

The residuals from a regression model are evaluated as the difference between the actual and the fitted values of the model. Each residual is a component of the associated observation which is not predicted. After selecting the regression variables and fitting a regression model, it is necessary to plot the residuals to check that the assumptions of the model have been satisfied.

3.4.1 DIAGNOSTICS - CONSTANT VARIANCE OF THE RESIDUALS

The first to check for residuals is homoscedasticity which is defined as the constant variance of the residuals. In order to obtain homoscedasticity we need to plot the residuals of the fitted reduced model against its predicted values. If the scatterplot does not show a specific pattern then we can deduce that there is a constant variance of the residuals. In figure 1 below we present a general example of a non-constant variance of residuals (exhibits a pattern) and in figure 2 we present the scatterplot of our data set.

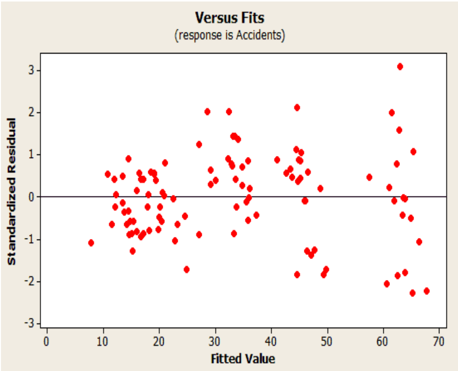

Figure 1 : Non – constant variance

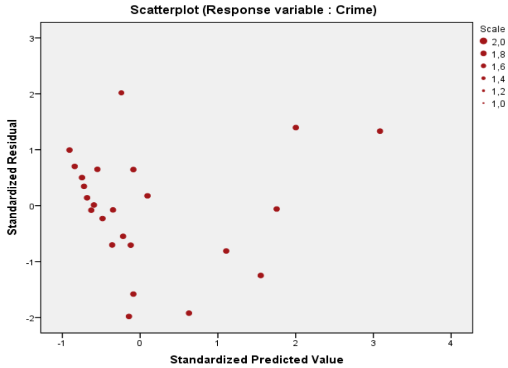

Figure 2 : Residuals VS Predicted (crime)

As we can observe from figure 2 when the residuals are plotted against the predicted values for our reduced model the data are randomly scattered and they do not exhibit any specific pattern, therefore the assumption of the constant variance of the residuals is valid.

3.4.2 DIAGNOSTICS - INDEPENDENCE OF THE RESIDUALS

The next assumption is that the residuals are not serially correlated from one observation to another meaning that the size of the residual for one case has no impact on the size of the residual for the next case. The Durbin-Watson Statistic is used to test for the presence of serial correlation among the residuals. The value of the Durbin-Watson statistic ranges from 0 to 4. As a general rule of thumb, the residuals are uncorrelated if the Durbin-Watson statistic is approximately 2. A value close to 0 indicates strong positive correlation, while a value of 4 indicates strong negative correlation. For our data set, the value of Durbin-Watson is 1.817, indicating no serial correlation.

3.4.3 DIAGNOSTICS – NORMAL DISTRIBUTION OF THE RESIDUALS

As we have already mentioned in the abstract the last assumption with respect to the residuals of a fitted regression model is normality. In order to check the normality of the residuals we perform two tests, the Kolmogorov-Smirnov test and the Shapiro -Wilk test. In both tests the null hypothesis states that the residual are normally distributed with the alternative one to support that residuals are not distributed by normal distribution. The results for these tests are available in table 6 below.

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	p-value	Statistic	df	p-value
Standardized Residual	,117	28	,200*	,975	28	,715

Table 6: Normality tests for residuals

If we take in mind the p-values (0.2 and 0.715 both greater than 0.05) of this test that is given in table 6, we cannot reject the null hypothesis which stands for the normality of the residuals. Therefore, the residuals are normally distributed.

3.4.4 INTERPRETATIONS AND CONFIDENCE INTERVALS

Most of the times, in the analysis of regression models it is reasonable to interpret the parameters of the model in physical terms. For this purpose, we discuss below the physical interpretation of the parameters of the reduced model.

$$\boldsymbol{y = -645258 + 0.059x_2 + 118673x_5}$$

$\hat{\beta}_2$: When the population of the European Union increases by one person, the crime increases by 0.059, given that the job vacancy remains unchanged.

$\hat{\beta}_5$: When the job vacancy in the European Union increases by one, the crime increases by 118673, given that the population remains unchanged. However, it seems surprising that there exists a positive correlation between job vacancy and crime rates. This correlation can be explained by the fact that high economic growth suggests higher job availability which cannot be sustained by the local workforce; therefore there is a flow of foreign workers coming from all over the world in each European country.

The last step in this report is the estimation of the confidence intervals. We estimate the 95% confidence intervals for the parameters of the reduced model. The estimated confidence intervals are available in table 7 below.

Model		95,0% Confidence Interval for B	
		Lower Bound	Upper Bound
1	(Constant)	-971629,836	-318886,116
	POPULATION	,053	,066
	VACANCY	59553,144	177792,496

Table 7: Confidence Intervals for the parameter

The interpretation of the confidence interval for $\hat{\beta}_5$ (vacancy) is that, if samples of the same size are drawn repeatedly from this population and a confidence interval is calculated for each sample, then 95% of these intervals should contain the real value of $\hat{\beta}_5$ in the interval (59553.144, 177792.496). The same interpretation can be given to all the parameters.

4. RESULTS, CONCLUSIONS AND RECOMMENDATIONS

Through our statistical study, we are able to draw some valid conclusions. We have observed that both the size of the population and job vacancy are variables, which are evidentially positively correlated to crime rates, whereas government expenditure on public order and safety, degree of urbanization and unemployment do not have a significant effect on crime rates. Consequently, we can conclude that countries of a smaller population, like Cyprus, have lower crime rates than countries of a larger population, like England. However, reducing the population of any large country is not sensible! So what could be a realistic method to prevent crime? We have seen that the higher the job vacancy is, the higher the crime rates are. The answer could again be another regression model that would result in the variables that affect job vacancy. As you can see, regression is the solution to multiple problems and probably the key to a better world.

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EUCLIDEAN DIVISION AND THE CONTINUED FRACTIONS, PERIODICITY AND IRRATIONAL NUNBERS

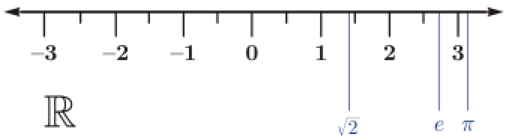
Claudia Dumbrava, Vasilis Hondros, Manolis Kyrios, Despina Maragou, Nikos Melissourgakis, Antonis Metallidis, Konstantina Mountaliani, Vasilis Niarhakos, Dimitris Oikonomakis, Giorgos Petsaggourakis, Paraskevas Politis, Maria Stratigou - Psarra, Pantelis Xagoraris
Zanneio Experimental Lyceum of Piraeus, Piraeus, Greece

ABSTRACT

In the textbook the irrational numbers are defined as the numbers that are not rational, that is the definition given by refusal. In our work we will present a description of the irrational number, avoiding the above denial. The description of rational and irrational numbers, based on the Euclidean Algorithm and Continued Fractions, intends to highlight that, despite the fact that quadratic irrationals are irrationals, they governed by a peculiar periodicity, a self-similarity, which was known in ancient times through geometry and is readily apparent today through the form of continued fractions. Therefore, the structure of quadratic irrationals resembles the one of fractals, which are also governed by self-similarity and thus they find frequent use in science and art.

INTRODUCTION

As we know in mathematics, real numbers are represented as points of a continuous straight line. Real numbers include all the rational numbers, namely, any number that can be expressed as the quotient of two integers *p, q with q different of zero* or fraction with numerator an integer *p* and denominator an non – zero integer *q, in symbols,* $r = \frac{p}{q},\ \ p, q \in \mathbb{Z},\ \ q \neq 0$, such as the integer −5 and the fraction 4/3, and all the irrationals, namely, any number **that cannot** be expressed as a ratio of integers, such as √2 (1.41421356…, the square root of two, an irrational algebraic number) and π (3.14159265…, an irrational transcendent number).



However, we notice that the irrational numbers are defined indirectly, via a negation. In our work we shall try to give an answer to the question of how we can define the irrationals without using the above negation. As we will show below, in ancient Greece, Pythagoreans and Plato's student Theaetetus, as Euclid has recorded in the books VII and X of his Elements, have distinguished the rational numbers (logos of numbers) and irrationals (logos of magnitudes) via Euclidean Algorithm (Anthyphairesis), and in the other hand, the theory of continued fractions, which has been developed by mathematicians of 18th century such as Euler, Lagrange, Gauss, makes the same distinction between the two kinds of real numbers. The description of rationals – irrationals in the above ways has the advantage to distinguish them directly and independently of one another, avoiding the indirect definition of irrationals as a denial of rationals.

RATIONAL NUMBERS AND EUCLIDEAN ALGORITHM

Firstly, a rational number may be described in several ways, i.e. in a fractional form either in a decimal form or by the Euclidean division. The theorem of Euclidean Division is the following, If α, β natural numbers with α>β>0, then there exist unique natural numbers π, υ with the following properties:

$$\alpha = \beta \cdot \pi + \upsilon \quad and \quad 0 \leq \upsilon < \beta$$

So, we give two examples of how we can write a rational number:

a) the division 4:5, then we have the fractional form $\frac{4}{5}$, the decimal form 0,8 which is derived by classical division

4	5
40	**0,8**
0	

b) 124:5, the fraction form is $\frac{124}{5}$, the decimal form is 24,8 derived by

124	5
24	**24,8**
4 0	
0	

and if we apply the Euclidean division we have $124 = 5 \cdot 24 + 4$, with π = 24, υ = 4<5.
Euclidean division is equality which implements the steps of the Euclidean Algorithm as described in the 7th book of the Euclid's Elements, which is attributed to Plato's student Theaetetus, where there are definitions and propositions which are concerned to the theory of proportions of numbers. Especially, the Propositions 1 and 2 (Πρότασις αʹ, Πρότασις βʹ) describe the finite Euclidean Algorithm (which, in ancient greek, is called Anthyphairesis) between two numbers a and b, where a>b:

Πρότασις αʹ	**Proposition 1**
Δύο ἀριθμῶν ἀνίσων ἐκκειμένων, ἀνθυφαιρουμένου δὲ ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε καταμετρῇ τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.	When two unequal numbers are set out, and the less is continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, then the original numbers are relatively prime.
Πρότασις βʹ	**Proposition 2**
Δύο ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.	To find the greatest common measure of two given numbers (which are) not prime to one another.

Wikipedia's Lemma 'Euclidean algorithm' describes the procedure of Euclidean Algorithm, which the above ancient text formulated, as following:
'The Euclidean algorithm proceeds in a series of steps such that the output of each step is used as an input for the next one. Let k be an integer that counts the steps of the algorithm, starting with zero. Thus, the initial step corresponds to $k = 0$, the next step corresponds to $k = 1$, and so on. Each step begins with two nonnegative remainders r_{k-1} and r_{k-2}. Since the algorithm ensures that the remainders decrease steadily with every step, r_{k-1} is less than its predecessor r_{k-2}. The goal of the kth step is to find a quotient q_k and remainder r_k that satisfy the equation

$$r_{k-2} = q_k\, r_{k-1} + r_k$$

and that have $r_k < r_{k-1}$. In other words, multiples of the smaller number r_{k-1} are subtracted from the larger number r_{k-2} until the remainder r_k is smaller than r_{k-1}.
In the initial step ($k = 0$), the remainders r_{-2} and r_{-1} equal a and b, the numbers for which the GCD (gratest common divisor) is sought. In the next step ($k = 1$), the remainders equal b and the remainder r_0 of the initial step, and so on. Thus, the algorithm can be written as a sequence of equations

$a = q_0\, b + r_0$

$b = q_1\, r_0 + r_1$

$r_0 = q_2\, r_1 + r_2$

$r_1 = q_3\, r_2 + r_3 \ldots$

If a is smaller than b, the first step of the algorithm swaps the numbers. For example, if $a < b$, the initial quotient q_0 equals zero, and the remainder r_0 is a. Thus, r_k is smaller than its predecessor r_{k-1} for all $k \geq 0$.
Since the remainders decrease with every step but can never be negative, a remainder r_N must eventually equal zero, at which point the algorithm stops. The final nonzero remainder r_{N-1} is the greatest common divisor of a and b. The number N cannot be infinite because there are only a finite number of nonnegative integers between the initial remainder r_0 and zero.'
The following examples illustrate the described finite procedure.

$13 = 5 \cdot 2 + 3$, $q_o=2$, $r_o=3$ $5 = 3 \cdot 1 + 2$, $q_1=1$, $r_1=2$ $3 = 2 \cdot 1 + 1$, $q_2=1$, $r_2=1$ $2 = 1 \cdot 2$, $q_3=2$, $r_3=0$; algorithm ends	$1071 = 462 \cdot 2 + 147$, $q_o=2$, $r_o=147$ $462 = 147 \cdot 3 + 21$, $q_1=3$, $r_1=21$ $147 = 21 \cdot 7$, $q_2=7$, $r_2=0$; algorithm ends
Gcd(13,5) = 1, So, 13 and 5 are relatively primes	Gcd(1071,462)=21 So, 1071 and 462 are not relatively primes
If we select the quotients of the Euclidean Algorithm, we have in each case	
13:5=[2,1,1,2]	1071:462=[2,3,7]

RATIONAL NUMBERS AND CONTINUED FRACTIONS

A **continued fraction** is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of *its* integer part and another reciprocal, and so on. In a **finite continued fraction** (or **terminated continued fraction**), the iteration/recursion is terminated after finitely many steps by using an integer in lieu of another continued fraction.
Euler in his 1737 work, De Fractionbous Continuis, showed that every rational number can be expressed as a finite simple continued fraction.
But how will we convert a rational number into a continued fraction? Let's take the examples 13:5 and 1071:462, which we examine in the previous paragraph.
13/5 is 2 lots of 5, with 3 left over, or, in terms of ordinary fractions:

$$\frac{13}{5} = \frac{5+5+3}{5} = 2 + \frac{3}{5}$$

The fraction $\frac{3}{5}$ is written as a fraction with nominator equal to 1 and denominator equal to the fraction $\frac{5}{3}$, so we have: $\frac{13}{5} = 2 + \frac{1}{\frac{5}{3}} = 2 + \frac{1}{\frac{3+2}{3}} = 2 + \frac{1}{1+\frac{2}{3}}$

We apply the same process to the fraction $\frac{2}{3}$ and have the result:

$$\frac{13}{5} = 2 + \frac{1}{1+\frac{2}{3}} = 2 + \frac{1}{1+\frac{1}{\frac{3}{2}}} = 2 + \frac{1}{1+\frac{1}{\frac{2+1}{2}}} = 2 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}} = 2 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}}$$

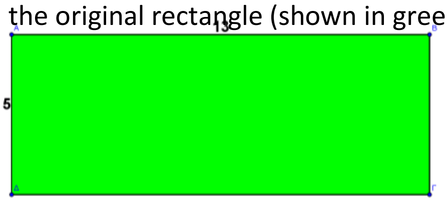
So, if we select the integers, which are in front of the expansion of finite continued fraction 13/5 then the rational number may be written as 13/5= **[2,1,1,2]**.
We notice that the process of finding the continued fraction expansion of the rational number $\frac{13}{5}$ is essentially identical to the process of applying the Euclidean algorithm to the pair of integers given by its numerator 13 and denominator 5.
Similarly, we have for the rational number 1071:462

$$\frac{1071}{462} = 2 + \frac{147}{462} = 2 + \frac{1}{\frac{462}{147}} = 2 + \frac{1}{3 + \frac{21}{147}} = 2 + \frac{1}{3 + \frac{1}{\frac{147}{21}}} = 2 + \frac{1}{3 + \frac{1}{7}}$$

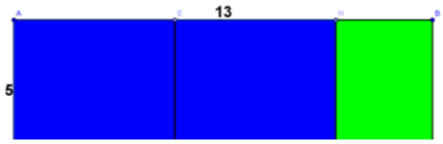
So, the rational number may be written as 1071/462= **[2,3,7].**

VISUALISING THE ABOVE PROCESS

According to the Wikipedia's Lemma 'Euclidean algorithm' we have:
'The Euclidean algorithm can be visualized in terms of the tiling analogy given above for the greatest common divisor. Assume that we wish to cover an a-by-b rectangle with square tiles exactly, where a is the larger of the two numbers. We first attempt to tile the rectangle using b-by-b square tiles; however, this leaves an r_0-by-b residual rectangle untiled, where $r_0 < b$. We then attempt to tile the residual rectangle with r_0-by-r_0 square tiles. This leaves a second residual rectangle r_1-by-r_0, which we attempt to tile using r_1-by-r_1 square tiles, and so on. The sequence ends when there is no residual rectangle, i.e., when the square tiles cover the previous residual rectangle exactly. The length of the sides of the smallest square tile is the GCD of the dimensions of the original rectangle.' For example, the smallest square tile in the adjacent figure is 1-by-1 (shown in purple), and 1 is the GCD of 13 and 5, the dimensions of the original rectangle (shown in green).

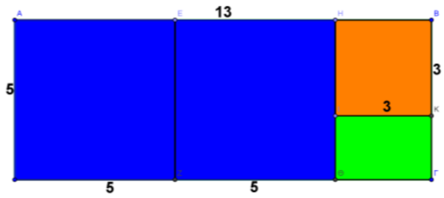


So, if we consider a 13-by-5 rectangle and apply step by step the above process, we have the following:



$13 \cdot 5 = (5 \cdot 5) \cdot \mathbf{2} + (3 \cdot 5)$, $q_o=2$, $r_o=3$

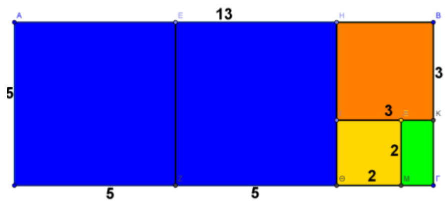
$\frac{13}{5} = 2 + \frac{1}{\frac{5}{3}}$



$13 \cdot 5 = (5 \cdot 5) \cdot \mathbf{2} + (3 \cdot 5)$, $q_o=2$, $r_o=3$

$3 \cdot 5 = (3 \cdot 3) \cdot \mathbf{1} + (2 \cdot 3)$, $q_1=1$, $r_1=2$

$2 + \frac{1}{1+\frac{2}{3}}$

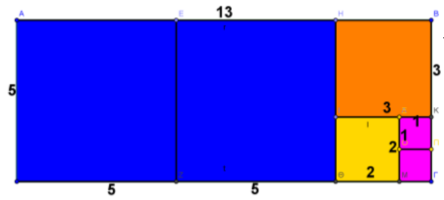


$13 \cdot 5 = (5 \cdot 5) \cdot \mathbf{2} + (3 \cdot 5)$, $q_o=2$, $r_o=3$

$3 \cdot 5 = (3 \cdot 3) \cdot \mathbf{1} + (2 \cdot 3)$, $q_1=1$, $r_1=2$

$3 \cdot 2 = (2 \cdot 2) \cdot \mathbf{1} + (1 \cdot 2)$, $q_2=1$, $r_2=1$,

$2 + \frac{1}{1+\frac{1}{\frac{3}{2}}}$



$13 \cdot 5 = (5 \cdot 5) \cdot \mathbf{2} + (3 \cdot 5)$, $q_o=2$, $r_o=3$

$3 \cdot 5 = (3 \cdot 3) \cdot \mathbf{1} + (2 \cdot 3)$, $q_1=1$, $r_1=2$

$3 \cdot 2 = (2 \cdot 2) \cdot \mathbf{1} + (1 \cdot 2)$, $q_2=1$, $r_2=1$

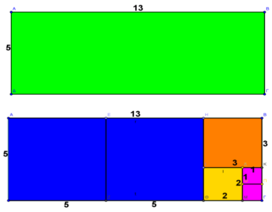
$2 = \mathbf{1} \cdot 2$, $q_3=2$, $r_3=0$ $\quad 2 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}}$

13:5=[2,1,1,2]

So, in the following scheme we summarize what we developed about rational numbers in the above paragraphs:

Visualization of $\frac{13}{5}$

$$13 = 5 \cdot 2 + 3$$
$$5 = 3 \cdot 1 + 2$$
$$3 = 2 \cdot 1 + 1$$
$$2 = 1 \cdot 2$$



$$\frac{13}{5} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

$$\frac{13}{5} = [2,1,1,2]$$

IRRATIONAL NUMBERS AND EUCLIDEAN ALGORITHM

The Euclidean Division (anthyphairesis) for magnitudes was presented by Euclid in the tenth Book of Elements, which is attributed to Plato's student Theaetetus, reflecting the Pythagorean tradition. The result of Euclidean Division between two magnitudes a and b (a>b) is obtained by the process of finding the common measure between of two magnitudes. This process is either finite (rational number a / b) or infinite (irrational number a / b). When, in the process of infinite case, periodicity is appeared (self-similarity) then the parts – magnitudes of division are quadratic irrationals – a fact which separates them from the other irrationals.

We will deal with the definitions α΄ and β΄ and proposition β΄ of the tenth book of the Elements of Euclid, which we present below:

Ορισμός α΄	**Definition 1**
Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μετρῳ μετρούμενα, **ἀσύμμετρα** δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.	Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.
Ορισμός β΄	**Definition 2**
Εὐθεῖαι **δυνάμει σύμμετροί** εἰσιν, ὅταν τὰ ἀπ᾽ αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, **ἀσύμμετροι** δέ, ὅταν τοῖς ἀπ᾽ αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι.	Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure.
Πρότασις β΄	**Proposition 2**
Εὰν δύο μεγεθῶν [ἐκκειμένων] ἀνίσων ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ καταλειπόμενον μηδέποτε καταμετρῇ τὸ πρὸ ἑαυτοῦ, ἀσύμμετρα ἔσται τὰ μεγέθη.	If, when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it, then the two magnitudes are incommensurable.

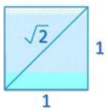
The above text (given, also, in ancient Greek) can be given to the following scheme:

Magnitudes a, b with a > b

Commensurable *Incommensurable*

Commensurable in square *incommensurable in square*

Then, we try to understand the above definitions through the following examples.
The case of commensurable magnitudes a, b, is like the case of the rational number a:b , which we have examined above.
The incommensurable magnitudes are divided into two types, the commensurable in square and the incommensurable in square. In the first type belong magnitudes as square roots of physical numbers, like the diagonal and side of square ($\sqrt{2}/1$), $\sqrt{3}/1$, etc., where the square of these magnitudes are physical numbers. In the second type belong magnitudes as pi (π), nth roots of physical numbers where n is a physical number and n>2 and the square of these magnitudes are not physical numbers.
The Euclidean Algorithm (anthyphairesis), in the case of commensurable in square magnitudes, has proven to be infinite having the property of periodicity, which means that the quotients of the divisor steps are repeated continuously after a divisor step. The property of periodicity gives the structure of self-similarity in these numbers. These magnitudes are quadratic irrationals. Consider the example of diagonal to side of the square:



Let a=1 is the side and d=$\sqrt{2}$ is the diameter of a square, correspondingly, $d^2 = 2 \cdot a^2$
We apply the euclidean algorithm to d:a. In every step we'll apply Euclidean division's pattern:

$$y = q \cdot x + r\,, \qquad r < x$$

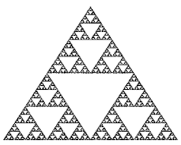
looking for the appropriate quotient q under the restrictions

$$d^2 = 2 \cdot a^2 \text{ and } r < x.$$

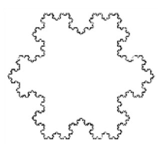
By the appropriate quotient, $y = q \cdot x + r$ will be true, otherwise false.
1st step: $d = \sqrt{2} = 1 \cdot q + r_1,\ r_1 < a = 1,$ true for $q = 1$
(because, $1 < \sqrt{2} < 2 \ \Leftrightarrow a < d < 2 \cdot a \Leftrightarrow a^2 < d^2 < 4 \cdot a^2 \Leftrightarrow 1 < 2 < 4$)
2nd step: $a = r_1 \cdot q + r_2,\ r_2 < r_1,$ false for $q = 1$, true for $q = 2$
and so on.
Finally, we get the following quotients: 1,2,2,…, which yield

$$\frac{d}{a} = \frac{\sqrt{2}}{1} = [1, \bar{2}]$$

Therefore, the structure of quadratic irrationals resembles the one of fractals, where fractal is a natural phenomenon or a mathematical set that exhibits a repeating pattern that displays at every scale and thus is governed by self-similarity. Below, we give two examples of fractals:



Sierpinski Triangle



Koch Snowflake

The Euclidean Algorithm, in the case of incommensurable in square magnitudes, has proven to be infinite without the property of periodicity. For example, if we consider the irrational number pi (π) and apply the Euclidean Algorithm then the quotients which we have are: $\pi = [3,7,15,1,292,1, \ldots .]$.

IRRATIONAL NUMBERS AND CONTINUED FRACTIONS

According to the theory of continued fractions, a quadratic irrational number is an irrational real root of the quadratic equation $ax^2 + bx + c = 0, a, b, c \in \mathbb{Z}$. It turns out that **every** square root has a continued fractions that ends up as a repeating pattern (infinite **periodic** continued fractions).
Euler (1707-1783) has proved that if *x* is a regular periodic continued fraction, then *x* is a quadratic irrational number and Lagrange (1736-1813) has proved if *x* is a quadratic irrational, then the regular continued fraction expansion of *x* is periodic.

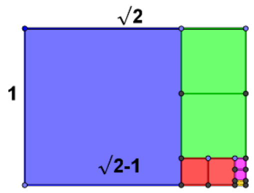
Expressing √2 as a continued fraction we have √2 is bigger than 1 thus we have √2 = 1 + 1/x then (√2 - 1) = 1/x then x = √2 + 1 (1) then x = 1 + 1/x + 1 = 2 + 1/x, so we the quadratic equation $x^2 - 2x - 1 = 0$ and the continued fraction:
$x = 2 + \frac{1}{x} = 2 + \frac{1}{2+\frac{1}{x}} = 2 + \frac{1}{2+\frac{1}{2+\frac{1}{x}}} = \cdots$ then by (1) we have $x - 1 = \sqrt{2}$, thus the continued fraction of √2 is $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}}$ or √2=[1,2,2,2,….]=$[1;\overline{2}]$.

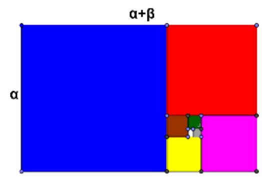
Another example is that of golden ratio which has the following definition:
two quantities are in the **golden ratio** if their ratio is the same as the ratio of their sum to the larger of the two quantities. The figure on the right illustrates the geometric relationship. Expressed algebraically, for quantities *a* and *b* with *a* > *b* > 0,
$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi,$ where the Greek letter phi (φ or ϕ) represents the golden ratio. Its value is:

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$

Number phi=$\frac{1+\sqrt{5}}{2}$ is the root of quadratic equation $x^2 - x - 1 = 0$ and the continued fraction is:
$x^2 = x + 1 \Leftrightarrow x = 1 + \frac{1}{x}, x \neq 0 \Leftrightarrow x = 1 + \frac{1}{x} = 1 + \frac{1}{1+\frac{1}{x}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{x}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}} = \cdots$or

phi=[1,1,1,1,…]= $[\overline{1}]$.
We notice that the continued fraction of two examples are infinite and have the property of periodicity, namely they have the property of self-similarity and therefore, we conclude that, by the construction of continued fraction also, the quadratic irrational numbers have the structure which resembles the one of fractals.
Visualizing the above examples we have:





In addition, it has been proved that the irrational which are not quadratic are represented by infinite continued fraction which they have not the property of periodicity, for example,

$$3 + \frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{292+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{14+\frac{1}{2+\frac{1}{1}}}}}}}}}}}}}}$$

$$e = 2 + \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{4+\frac{1}{1+\frac{1}{1+\ddots}}}}}}}$$

π=

In concluding one has that the set of real numbers are divided into two sets, the set of rational numbers and the one of irrational numbers. The rational number a/b is represented by finite Euclidean algorithm of a to b ending at gcd (a,b) and is expessed as a finite continued fraction, with the quotients of Euclidean Algorithm a to b appearing in it. The irrational numbers are divided to quadratic irrational and not quadratic irrational, where the first one are represented by infinite periodic Euclidean algorithm and are expressed as infinite periodic continued fraction and have the property of self-similarity while the second one are represented by infinite Euclidean algorithm and are expressed as infinite continued fraction and have not the property of self-similarity.

APPLICATIONS

An application of the construction of √2 is the ISO 216, which specifies international standard (ISO) paper sizes used in most countries in the world today. The standard defines the "A" and "B" series of paper sizes, including A4, the most commonly available size. Two supplementary standards, ISO and ISO 269, define related paper sizes; the ISO 269 "C" series is commonly listed alongside the A and B sizes. All ISO 216, ISO 217 and ISO 269 paper sizes (except DL) have the same aspect ratio $1{:}\sqrt{2}$. This ratio has the unique property that when cut or folded in half widthwise, the halves also have the same aspect ratio. Each ISO paper size is one half of the area of the next larger size.

Another application is that of golden section. Many buildings and artworks have the Golden Ratio in them, such as the Parthenon in Greece, but it is not really known if it was designed that way. Many painters, also, like Leonardo da Vinci, Salvador Dali, Mondrian have used the golden section extensively in his paintings. In nature found the golden ratio expressed in the arrangement of parts such as leaves and branches along the stems of plants and of veins in leaves. Musicians, such as Erik Satie and Béla Bartók have used the golden ratio in several of their pieces.

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COLORFUL PLANES

Wojciech Niewiara*, Piotr Sikorski, Michał Stach*, Tobiasz Szemberg****
*VII LO im Zofii Nałkowskiej, Skarbińskiego 5, 30-071 Kraków, Poland
**Gimnazjum im. Jana Matejki w Zabierzowie, ul. Kolejowa 15, 32-084 Zabierzów, Poland

ABSTRACT

In the present note we study some generalizations of a problem taken from the mathematical olympiad for medium schools in Poland in 2013. This leads to a coloring problem on a finite plane. We are not able to solve the problem completely but we present a partial solution and hope that the readers will find the problem interesting and challenging.

1. INTRODUCTION

There is a number of problems revolving around the idea of coloring the plane. One of the most famous solved problems is that of four colors: can every map (partition of plane into subsets - countries) be colored by only four colors in such a way that regions sharing a common boundary (other than a single point) are colored in different colors? This problem has been solved in the 70's of the XX century. Another famous, yet this time unsolved problem was formulated by Hadwiger and Nelson [1].

Problem 1.1.
What is the least number of colors needed in order to color the plane in such a way that two arbitrary points, whose distance is 1, have different colors?
It is striking that an answer to this problem is not known! It is known only that the correct answer is one of the numbers 4, 5, 6 or 7.
Of course, we will also not answer the above question here. However we will consider a problem of similar taste. The inspiration come from a problem in the Mathematics Olympiad for Medium Schools in Poland in 2013. We recall first the problem.

Problem 1.2.
Paint a plane in such a way that every line in the plane has points in at most two different colors. What is the highest possible number of colors, which can be used to paint the plane. Justify the answer.
This problem opens door to a number of possible generalizations. We mention here just three of them:

a. painting of a higher dimensional space;
b. allowing a higher number of colors on each line;
c. considering the same problem on different kinds of planes (not necessarily the euclidean plane).

We will be most interested in variants b. and c. Let us mention just briefly the solution to variant a. In the 3-dimensional space it goes as follows. We paint the whole space with the exclusion of one plane with color number 1. Then we paint the plane left blank with exception of a single line with color number 2. Similarly, we paint the remaining line with exception of a single point with color number 3 and finally we paint the last blank point with color number 4. This pattern can be easily generalized to the space of dimension n, where one can use n+1 colors to paint the space in such a way that every line in that space has points in at most two different colors.

2. PROBLEM 1.2 WITH THE HIGHER NUMBER OF COLORS

We will show now a somewhat surprising solution to the following modification of Problem 1.2.

Problem 2.1.

Paint the plane in such a way that every line in this plain has points on at most 3 colors. What is the maximal number of colors which can be used?

It turns out that in this situation one can use infinitely many (even more than countably many) colors! The argument is very easy but it took us a while to realize that the answer to problem 2.1 is not a finite number. Here comes the argument. Let C be a smooth curve of degree 2 in the plane, for example a circle. Then C has at most two common points with any line in the plane. Hence, if we paint all plane but C with one color and use mutually distinct colors for all points on C, then we will use (uncountably) infinitely many colors and the condition of the problem will be satisfied: every line, apart of the color of the background (i.e. the plane without C) has at most two points (the intersection points with C) in different colors.

This leads to another possible generalization.

Problem 2.2.

Paint the plane in such a way that every curve of degree d contains points in at most m colors. What is the highest possible number of colors c(d,m) which can be used for the painting of this kind?

Thus c(1,2)=3 and c(1,3) equals infinity by what we said before. We hope to come back to this extremely interesting (and challenging) question in the next future.

3. PROBLEM 1.2 ON A FINITE PLANE

This is the core part of our paper. We consider now a finite plane, i.e. a plane which consists of finitely many points. Since Descartes (1596-1650) it is known that one can identify points in the plane with pairs of numbers (coordinates of these points). The usual Euclidean plane consists of infinitely many points, these points are indexed by pairs of real numbers, which are, of course, also infinite in number. Thus a finite plane must correspond to numbers in a finite set, in which one can define the addition and multiplication. We will consider here by the way of an example the set of division rests of integers by division modulo 5, i.e. the set

$$F_5=\{0,1,2,3,4\}.$$

Remark 3.1.

Taking the number 5 here has some practical motivation. The plane which we will define will have enough many points to allow interesting geometry and enough few points to allow simple pictures illustrating what is going on. An interested Reader is invited to repeat our considerations for the number 7. An advanced Reader should be able to repeat them for an arbitrary prime number.

In the set F_5 the arithmetic operations are defined modulo 5. That means for example that

1+4=0 because 5=1*5+0,

3+3=1 because 6=1*5+1,

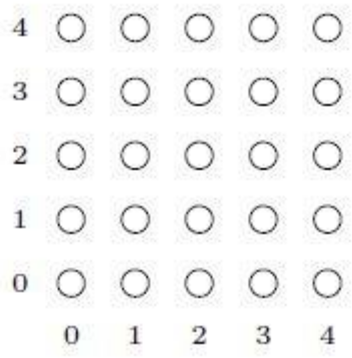
2-4=3 because -2=(-1)*5+3,

4*4=1 because 16=3*5+1,

3:4=2 because 2*4=8=1*5+3,

4:3=3 because 3*3=9=1*5+4.

Coming back to the geometry, we define the plane E=F_5 x F_5 as the set of pairs (a,b) with a,b division rests modulo 5. This plane can be schematically indicated as follows:



The original Problem 1.2 has an unaltered solution in the case of finite plane E. We have presented already above a scheme showing that one can use at least 3 colors to paint the plane so that the conditions of Problem 1.2 are satisfied. Now we explain that a higher number of colors is not possible. Let us assume to the contrary that there were used 4 different colors. Assume that the points P, Q, R and S are painted in pairwise distinct colors. No 3 of these points can be collinear because this would contradict the condition that there are at most 2 colors on every line. Hence there are three distinct pairs of lines PQ and RS, PR and QS, PS and QR. At least one of these pairs is not a pair of parallel lines, so that it intersects in a point X. Without loss of generality we may assume that PQ and RS is the pair of intersecting lines. Then X would have to be either in the color of P or Q, because it lies on the line PQ. On the other hand, it lies on the line RS, so it would have to be in the color of R or S. But all these 4 colors are different. This gives a contradiction and thus finishes the solution of Problem 1.2.
The modified problem stated in Problem 2.1 has in the case of a finite plane, surely a finite answer which is at most 25, since there are only 25 points in the plane E. Hence the solution proposed for the Euclidean plane will not work here. Of course, it is not possible to color every point on E in a different color, since then there would be points in 5 different colors on a line. So the question is: how many colors can be used in the finite plane E?

3.1 UPPER BOUNDS ON THE NUMBER OF COLORS.

We begin with a very naive bound, which motivates further refinements.

Lemma 3.2.
It is not possible to color the plane E= F_5 x F_5 with 16 or more colors in such a way that every line in this plane has points in at most three distinct colors.

Proof. We will apply the pigeonhole principle. Of course it is sufficient to show the claim for exactly 16 colors. Let us assume to the contrary that there exists a painting with 16 different colors satisfying the conditions of Problem 2.1. This implies that there are 16 points in E, which are painted in 16 mutually distinct colors. Let us consider a pencil of parallel lines, for example the vertical lines given by equations x=conts with const an element of F_5. There are 5 lines in this pencil. The pigeonhole principle implies then that there exists at least one line in the pencil containing 4 from the selected 16 points. The conditions of Problem 2.1 fail for this line. This contradiction ends the proof.
We will use the next Lemma in order to improve the bound given in Lemma 3.2.

Lemma 3.3.
If the plane E has been painted in such a way that the conditions of Problem 2.1 are satisfied and there exist 3 parallel lines which contain points in three different colors and all these nine colors are different, then the plane was painted with exactly 9 colors.

Proof. Let X be a point not lying on one of the three parallel lines. Then a line passing through X and not parallel to the three given lines intersects each of them in a point and these 3 intersection points have pairwise distinct colors. This means that X is painted in one of these 3 colors. Hence the whole plane is painted with 9 colors.

Lemma 3.4.
It is not possible to paint the plane E= F_5 x F_5 using 13 (or more) colors in such a way that all lines in the plane contain points in at most 3 different colors.

Proof. Similarly as in the proof of Lemma 3.2 we will use the pigeonhole principle. Let us assume to the contrary that a painting with 13 colors exists. Let us consider an arbitrary pencil of parallel lines, for example the vertical lines. These lines cover the whole plane, so they contain points in 13 different colors. Since each of them contains points in at most 3 different colors, there must be 3 lines in this pencil, which contain points in exactly 3 different colors and the 9 colors altogether are mutually distinct. Then the contradiction follows from Lemma 3.3 and we are done.

3.2 LOWER BOUND FOR THE NUMBER OF COLORS.

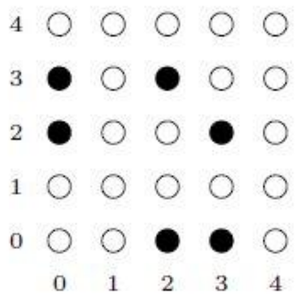
In order to provide a lower bound we will use the idea of the euclidean solution of Problem 2.1. To this end we need to consider curves of degree 2 on the plane E. Each such curve is given by the quadratic equation

(1) $ax^2+by^2+cxy+dx+ey+f=0$,

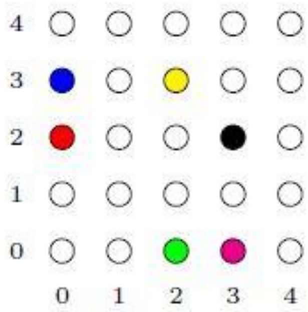
Where a,b,c,d,e,f are in F_5 and the equation is quadratic i.e. at least one of the numbers a,b or c is non-zero. Moreover, we are interested in conics which do not contain a line. Thus, we are for example not interested in the equation xy=0, since it describes the union of two lines: the x and the y axes. It turns out (see [2]) that if the set of solutions of equation (1) contains no line, then it consists of at most 6 points. The equation

$$x^2+y^2+xy+1=0$$

has exactly 6 solutions. This is indicated graphically in the figure below.



The picture below shows then a possible painting of the plane E with 7 colors (including the white color of the background points), which satisfies the conditions stated in Problem 2.1.



We expect that this is in fact the optimal solution, i.e. we expect that one cannot use 8 or more colors. Unfortunately, for the time being, we are not able to prove this. We wrote a computer program which checks all possible paintings case by case but we estimated that its running time is roughly 2 to the power 21. Hence it makes more sense to search for a traditional proof.
We summarize our results in the following Theorem.

Theorem 3.5.
There exists a painting of the plane E satisfying conditions of Problem 2.1 with 7 colors. The maximal number of colors which can be used for such a painting is at most 12.
Thus the solution to Problem 2.1 in the plane F_5 is a number from the set

$$\{7,8,9,10,11,12\}.$$

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GAME THEORY

***Ivan Barta, **Fran Ilčić, ***Andrija Mandić**

*XV. gimnazija, Peruanska 14 , Zagreb, Croatia, ivanbarta1234@gmail.com
**XV. gimnazija, Bukovačka 20a, Zagreb, Croatia, ilcic.fran@gmail.com
*** XV. gimnazija, IV. Bizek 7a, Zagreb, Croatia, mandic.andrija@yahoo.com.hr

ABSTRACT

Games are a type of social phenomena that are present in our everyday lives and from which we learn a lot and also have fun. But there is a lot of mathematics behind the mechanics of games and success in them.

Probability is, for one, a large determining factor in many games, such as card games. In some games the best option always exists, but it is hard to realise which one is it and why it is the best one. Such a situation is called the Nash equilibrium. In Nash equilibrium a player's strategy does not depend on other players' strategies and he cannot benefit from playing any other strategy, i.e. it is always the best one.

We have studied the principles of game theory and learned how to apply them in everyday life.

INTRODUCTION - MATHEMATICAL GAMES

Game theory is a field in mathematics which studies mathematical games. Mathematical games are processes in which we can define rules, strategies and outcomes by strict mathematical parameters. Game theory is used in decision making processes and can be applied not only to games but to economy, politics and everyday life as well.

A simple example is the rock-paper-scissors game. There are two players and each of them displays one simbol at the same time (rock, paper or scissors). Rock beats scissors, scissors beat paper and paper beats rock. The outcome is one point for the player who won and 0 for the other, or 0 for both if there is a draw. Players may have strategies for the game, they may choose to always play one of the simbols, or to randomly select one of the simbols, giving each of them a probability value that can be based on previous tries.

In our project we worked with simpler games with only two choices for each player to examine the basic principles and situations. To show the choices and the outcomes, game theory uses payoff matrices. This is an example of a simple mathematical game with two players, Annie and Kate. Annie can move up or down and Kate can move left or right. They get an amount of points accordingly (as shown by the matrix below) and the goal for both of them is to get as many points as possible.

		Kate			
		Left		Right	
Annie	Up	2	-2	1	-1
	Down	-3	3	0	0

Picture 1 - Simple payoff matrix

The red numbers are Annie's and the yellow Kate's points earned for specific combination of their moves.

STRATEGIES

A strategy is a way in which a player determines which choice should he make. If a player chooses one move and makes that move every time the game is played, until he changes his mind, he is using a pure strategy. If he chooses a random move every time the game is played, each move having a certain probability of being chosen, then he is using a mixed strategy. If a player is using a pure strategy, but changes it periodically, that can also be interpreted as if he is using a mixed strategy of those moves.

In our example with Annie and Kate mixed strategies can be defined with one parameter for each of them. Annie has probability p to move up, and so the probability of her moving down is $1\text{-}p$. Kate moves left with a probability q and right with $1\text{-}q$.

		Kate			
		Left *q*		**Right *1-q***	
Annie	**Up *P***	**3**	**-3**	**0**	**-1**
	Down *1-p*	**-2**	**2**	**1**	**0**

Picture 2 - Mixed strategies

Another important value is the expected payoff. The expected payoff of a move is the average number of points a player will get by making that move, given an opponent's mixed strategy. For moving up Annie will recieve 3 points q of the time and 0 points $1\text{-}q$ of the time. Same for down with -2 and 1 points So expected payoffs for Annie's moves up and down are calculculated like so:

$$U = q \cdot 3 + (1\text{-}q) \cdot 0 = 3q$$
$$D = q \cdot (-2) + (1\text{-}q) \cdot 1 = -3q+1$$

Following the analogy, expected payoffs for Kate's moves are:

$$L = -5p+2$$
$$R = -p$$

Both player's goals are to have as many points as possible, so they want to play the strategy which will give them the most points. This strategy is, of course, dependant on the opponent's strategy and is called the best response. The best response is one of the strategies that will give the player the highest expected payoff. Annie's best response will be to move up if $U>D \rightarrow 3q>-3q+1 \rightarrow q>1/6$, or to move down if $D>U \rightarrow q<1/6$. If Kate's q is exactly 1/6 Annie can play any strategy she wants because all of them will give her the same amount of points.

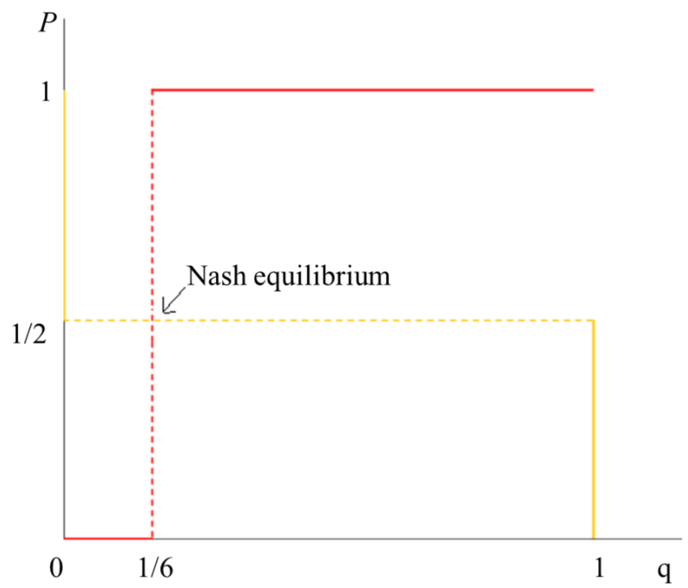
NASH EQUILIBRIUM

Nash equilibrium is a solution concept of a non-cooperative game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. So if both players were to change their strategies the outcomes could become profitable but if only one player's strategy was to change that player wouldn't profit of it.

Nash equilibrum was named after John Forbes Nash, Jr. , who proved that in every game there is at least one equilibrum in mixed strategies.

So how do we find Nash equilibrium in a mixed strategy game? Well it's simple for this kind of two player games.

As we mentioned before, when U>D Annie always plays up, and when D>U she always plays down. When U=D she can do anything, but when she decides what to do, next time Kate will change her *q* to get her highest payoff and so U will no longer be equal to D. Same goes for Kate's L and R and Annie's *p*. But what if in the start U equals D and L equals R? Let's look at that game. From U=D and L=R we know $p=\frac{1}{3}$ and $q=\frac{1}{2}$. Let's see what happens when players try to change their strategy. If Annie changes her strategy, her average outcome can't get any better, as we explained before (playing up or down average outcome is the same).That's because any strategy she plays is a best response. The same thing goes for Kate, so this point of the game is a Nash equilibrium.



Picture 3 - Graph of Annie's best move (p) dependance on q - red and Kate's best move (q) dependance on p - yellow.

Usually their strategies will not be the equilibrium ones, so they will keep changing them after every move to adjust to the opponent's average strategy until the present moment - the strategy they thing the opponent is using from the beggining. It was also proven that if they keep doing just that, their strategies will change periodically and it will look like they have been playing the equilibrium strategies all along. That is true for all games with an equilibrium.

PRISONERS' DILEMMA

A common problem in game theory is the prisoner's dillema. It is a situation with two prisoners that are put in seperate cells, interrogated simultaneously, and offered deals for betraying their fellow criminal. They have to choose if they want to betray their partner in crime. If both were to choose to betray the other both get 1 year less in prison. If neither betrays they get 3 years less in prison each. However, if one betrays the other and the other doesn't betray him he gets 5 years less in prison, while the betrayed one gets 0.

		Kate			
		Cooperate		Betray	
Annie	Cooperate	3	3	0	5
	Betray	5	0	1	1

Picture 4 - Prisoners' dilemma

So, what is the smarter choice? Well if a prisoner chooses to betray, the other is better of betraying because he gets 1 years less instead of 0. If a prisoner chooses not to betray, the other is once again better of betraying because he gets 5 years less instead of 3. So both prisoners will always choose to betray each other. However if they were both to cooperate with each other each of them would get 2 years less in prison than they do by betraying each other. This means that they would both profit by setting a deal to cooperate before the interrogation. However, both would still have the tendency to betray the other to profit even more and so they would most likely end up betraying each other again.

Would you betray your partner?

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https://en.wikipedia.org/wiki/Game_theory

THE EXPEDITION THAT BRIDGED THE GAP BETWEEN MATERIALS SCIENCE AND MATHEMATICS

Kolosnjaji Marijana and Govlja Tatjana
Primary and secondary school "Petro Kuzmjak", 25233 Ruski Krstur, Serbia,
Supervisors:
Natalija Budinski, Primary and secondary school "Petro Kuzmjak", Ruski Krstur, Serbia
dr. Mirjana Jovicic, Faculty of Technology, University of Novi Sad, Serbia

ABSTRACT

In this paper we are showing how mathematics can be applied in materials science. We observe mathematical models for energy of activation which is important parameter in chemical processes. We have described how can activation energy be mathematically determined from the real-world data. This data have been taken from the materials science scientific experiments and real-world.

INTRODUCTION

Materials science is an interdisciplinary field which studies the properties of materials and their application in many areas of science and engineering. It covers understanding of how materials can be created, used, changed and improved and how they react under different conditions - under different mechanical influences, changes in temperature, pressure. This specific branch of science is very useful and important because we constantly use materials in our everyday life. Materials are also the basis of high technology. They are used in biomedicine, production of computers, televisions, solar panels and many more items. Materials science incorporates elements of physics, chemistry and mathematics.

MATHEMATICAL REPRESENTATION OF ACTIVATION ENERGY

When materials scientists observe properties of materials, they observe many different parameters, one of which is activation energy. Activation energy represents the minimum of energy that needs to be provided to the reactants in order to start a chemical reaction. Graphical representation of activation energy in chemical reaction is displayed in the Figure 1.

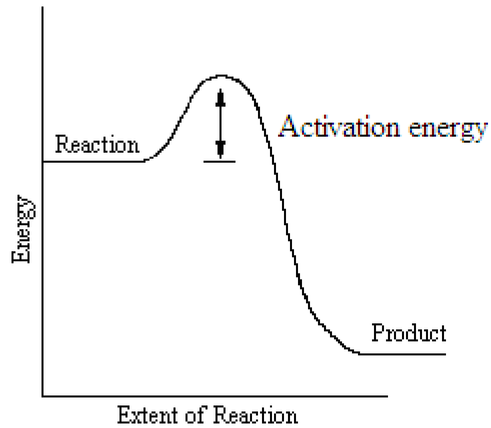

Figure 1. Activation energy of a chemical reaction

Mathematically, activation energy can be described using Arrhenius equation, a mathematical model that describes the dependence of the rate constant of a chemical reaction on temperatures and activation energy. The mathematical form Arrhenius equation is:

$$k = A \cdot e^{-\frac{Ea}{R \cdot T}}$$

K represents the rate constant of a chemical reaction [$mol^{1-n}(dm^3)^{n-1}s^{-1}$, *n is reaction order*], R represents the universal gas constant [8.314 J/Kmol], A is the pre-exponential factor, T represents the temperature [K] and Ea is the activation energy of a chemical reaction [J/mol].

Since Arrhenius equation is in an exponential form, it can be transformed to a linear form with the use of logarithm to be more suitable for calculating activation energy. The transformation of exponential form of Arrhenius equation to a linear form with the use of the properties of logarithms is showed in the Table 1

$k = A \cdot e^{-\frac{Ea}{R \cdot T}}$	Exponential form of Arrhenius equation
$\ln k = \ln A \cdot e^{-\frac{Ea}{R \cdot T}}$	$a = b \Rightarrow \ln a = \ln b, a, b > 0$
$\ln k = \ln A + \ln e^{-\frac{Ea}{R \cdot T}}$	$\ln ab \Rightarrow \ln a + \ln b, a, b > 0$
$\ln k = \ln A - \frac{Ea}{R \cdot T} \ln e$	$\ln a^b \Rightarrow b \ln a,\ a > 0$
$\ln k = \ln A - \frac{Ea}{R \cdot T}$	$\ln e = 1$
$\ln k = \ln A - \frac{Ea}{R} \cdot \frac{1}{T}$ (1)	Linear form of Arrhenius equation

Table 1 Mathematical transformation of Arrhenius equation

In the obtained linear form of the equation, the $\ln k$ represents the dependent variable, while $\frac{1}{T}$ is the independent variable of this linear equation. The point at which the line crosses the y-axis, otherwise known as the y-intercept is determined with $\ln A$. The slope or gradient of the line is determined with $-\frac{Ea}{R}$. Using this equation, activation energy can be determined knowing the slope of the line and the value of the universal gas constant.

MATHEMATICAL MODELS IN THE MATERIALS SCIENCE

Materials scientists use more sophisticated models for determining activation energies, such as Kissinger (Kissinger, 1957). The mathematical form of Kissinger model is:

$$\ln\left(\frac{\beta}{T_p{}^2}\right) = \ln\left(\frac{R \cdot A}{E_a \cdot g(\alpha)}\right) - \frac{E_a}{RT_p} \qquad (2).$$

In (2) β represents the heating rate [K/min] and $g(\alpha)$ is the function of the chosen model. This function is not being explicitly expressed. The Tp in the model (2) represents the temperature of the maximum speed of the reaction. Both temperatures are measured in Kelvins. These models are based on Arrhenius equation, and with their help, activation energy can be determined by measuring temperature under controlled heating rates in complex materials science experiments.

These mathematical models are transformed to a linear form with the use of simple mathematical operations to make the calculation of activation energy easier. It is shown in the example of Kissinger model (3). The expression $\ln\left(\frac{\beta}{T_p{}^2}\right)$ here represents the dependent variable, $\frac{1}{T_p}$ represents the independent variable and $-\frac{E_a}{R}$ is the slope of the line. Let us note the slope of the line as $a = -\frac{E_a}{R}$. The slope of the line can be determined knowing the values of the dependent and independent variable. Activation energy can be determined knowing the slope of the line and the value of the universal gas constant as follows

$$E_a = a \cdot R \quad (4).$$

APPLICATION OF THE MATHEMATICAL MODELS IN MATERIALS SCIENCE PRACTICE OF WOOD INDUSTRY

The application of mathematical models in practice started with the visit to the Faculty of Forestry at University of Belgrade and Faculty of Technology at University of Novi Sad where we had a chance to talk to scientists who work in the field of materials science and mathematics. We have participated in the measurements, and we collected the real-world data from the experiment. The goal of our expedition was to research about connection between mathematics and real world problems. The process of solving real world problems with the application of mathematics is usually called mathematical modelling (Budinski et all, 2015).

The experiment that we participated in was about examining how different types of wood affect the properties of Urea formaldehyde resin (UF), which is often used in the wood industry. Different types of woods were used for the comparison – beech, fir, poplar. There were made four samples. One sample was pure UF resin, the second sample was UF combined with beech, the third sample was UF combined with fir and the fourth sample was UF resin combined with poplar. These samples were processed under the different heating rates and the temperature of the maximum speed of the reaction was measured. The obtained results are presented in the Table 2.

Heating	**Temperatures of the maximum speed of the reaction [C°]**			
	UF resin	**UF resin + beech**	**UF resin + fir**	**UF resin + poplar**
5	85,6	88,6	87,1	98,4
10	95	97,5	95,6	108,5
15	101,3	103,2	101,9	113
20	105	107,2	106,1	117,2

Table 2. The results of the measurements in the scientific experiments (Popovic, 2011)

GEOGEBRA MATHEMATICAL MODELS OF MATERIALS SCIENCE PROBLEMS

Using GeoGebra (www.geogebra.com), we have constructed the graphs of Kissinger- model for the data presented in the Table 2. To make this model, we observed the values of natural logarithm of $\ln\left(\frac{\beta}{T_p^{\ 2}}\right)$, which in this model represents the value of the dependent variable, and values of $\frac{1}{T_p}$ which represents the independent variable. The obtained models are shown in the Figures 2, 3, 4, 5.

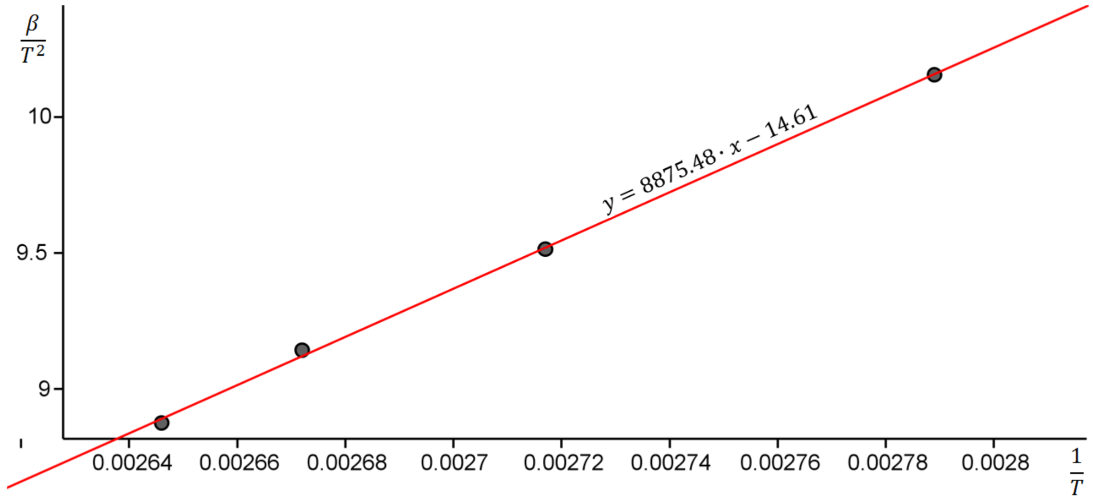

Figure 2. The Kissinger model of Urea formaldehyde resin

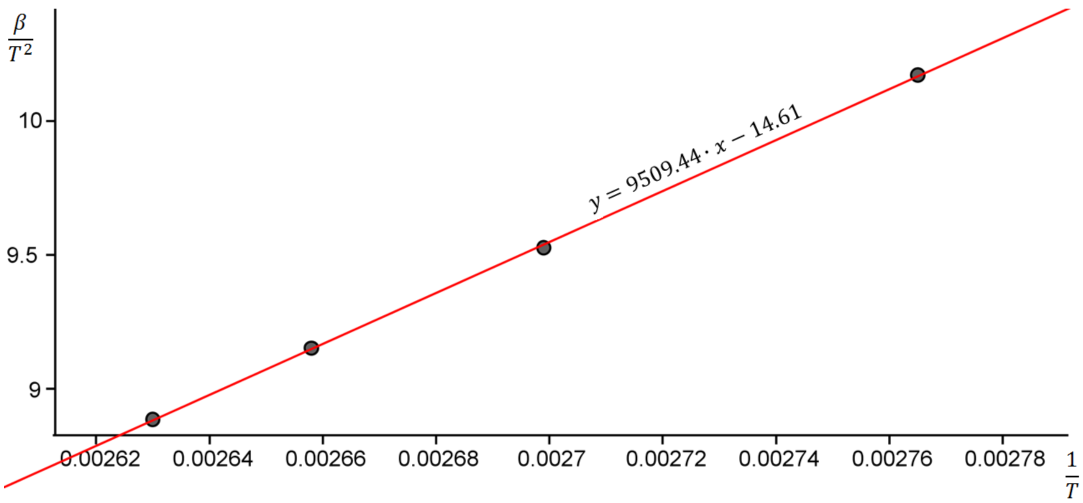

Figure 3. The Kissinger model of Urea formaldehyde resin in combination with beech

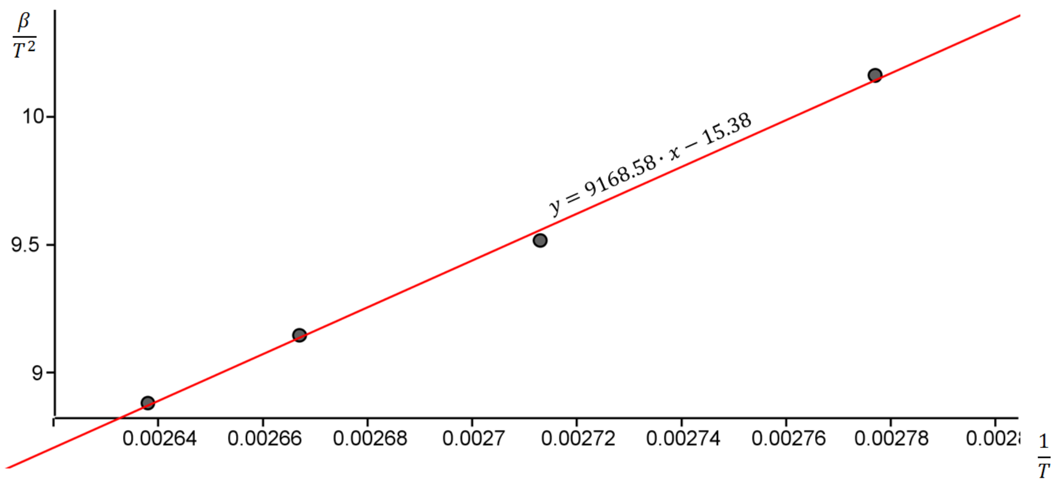

Figure 4. The Kissinger model of Urea formaldehyde resin in combination with fir

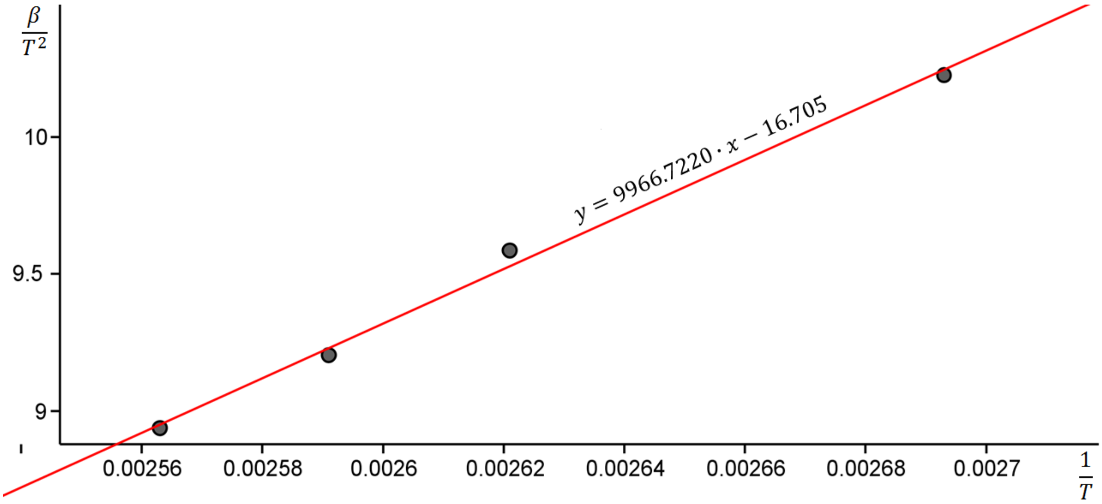

Figure 5. The Kissinger model of Urea formaldehyde resin in combination with poplar

By using options available in Geogebra, we received the value of the slope of the line for all samples. Knowing that the slope of the line in this model is (4) the value of the universal gas constant is 8.314 J/Kmol, we determined the activation energy. We repeated the process for other samples and we got the following values that are showed in the Table 3.

Table 3. The results of calculations

	T(UF resin)	T(UF resin + beech)	T(UF resin + fir)	T(UF resin + poplar)
a	8859.35	9509.44	9168.58	9966.72
$E_a\left[\frac{J}{mol}\right]$	73755.20	79023.48	76190.89	82823.45

CONCLUSION

The final conclusion is that there is no meaningful difference of the activation energies for the observed samples. That means that the usage of UF resins with the combination of wood particles provide the same performance as the UF resin alone. Mixing UF resin with the wood particles can diminish the cost of the production and save the environment, as well.

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www.geogebra.org

MATHEMATICS OF BIOLOGICAL EVOLUTION

De Chiara Emanuela *, di Virgilio Elena Maria,**
Scerbo Francesco ***
*student Liceo Scientifico "L. Siciliani" Catanzaro, Italy dechiaraemanuela@gmail.com
** student Liceo Scientifico "L. Siciliani" Catanzaro, Italy elenuzza98.EdV@gmail.com
***teacher Liceo Scientifico "L. Siciliani" Catanzaro, Italy fscerbo@libero.it

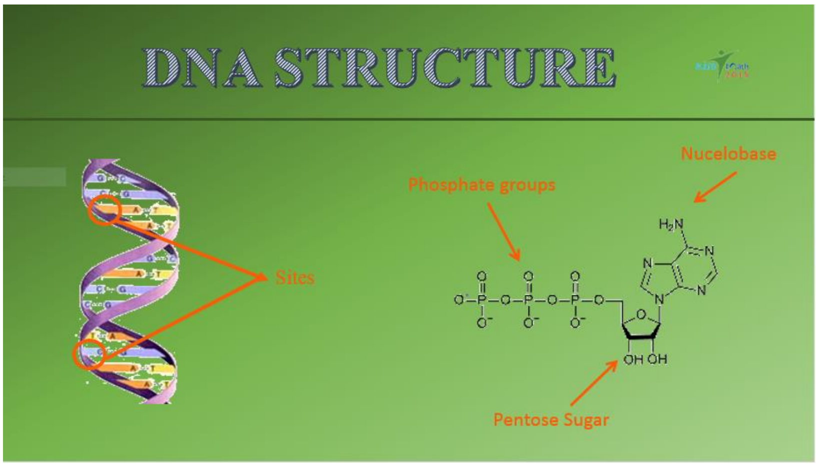
ABSTRACT

Aim of our work is to reproduce bioinformatics procedures using simple mathematics and DNA in order to build a phylogenetic tree using and comparing the nucleobases of different primates. We did this using the Jukes-Cantor mathematical model and finally comparing our results with those made by Mega6.

INTRODUCTION ABOUT DNA

DNA is well known as deoxyribonucleic acid and provides all the necessary genetic information for the development and the right working of living organisms.
5-. It has a double helix structure, that can be seen, more simply, as a spiral staircase where pegs are made of four nucleobases. These are: adenine that binds with thymine and guanine that binds with cytosine. The name we use to mean the exact position of the bases is site. The railing, instead, is made of a pentose sugar and a phosphate group.

NATURAL EVOLUTION



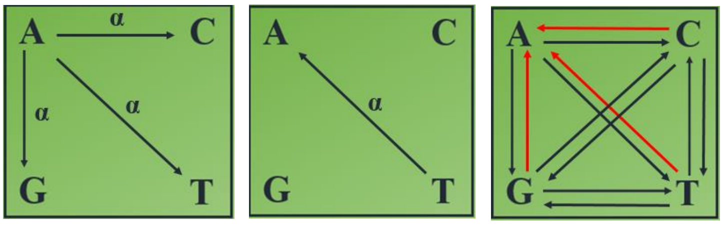
In each generation, from the union of two individuals, particular changes can occur. These changes are defined as mutations, that are spontaneous or induced and modify the genes' structure by replacing a nucleobases' couple. The combination of a set of male chromosomes and a female one cause the birth of a new organism. DNA mutations' effect during the subsequent generations of individuals defines the evolution of the species. To better explain this statement we will use the Jukes-Cantor mathematical model.

JUKES-CANTOR MODEL

Is the simplest mathematical model that describes these mutations during time. Our fundamental assumptions are:

- All the sites evolve independently
- All the sites change with the same probability
- All the substitutions are equally probable
- Substitution's speed is changeless in time
- Bases' composition is changeless

To symbolize mutation rates let's put nucleotides in the corners of a square. Each nucleotide can mutate in three different equiprobable ways i.e. with a probability of 3α.
However another base will only mutate to an «A» with a probability of α. If we look at all the possible mutations simultaneously there are only three arrows pointing to "A" for a total of 12. This means that we expect 3/12 = ¼ = 25% of the nucleotides to consist of "A".



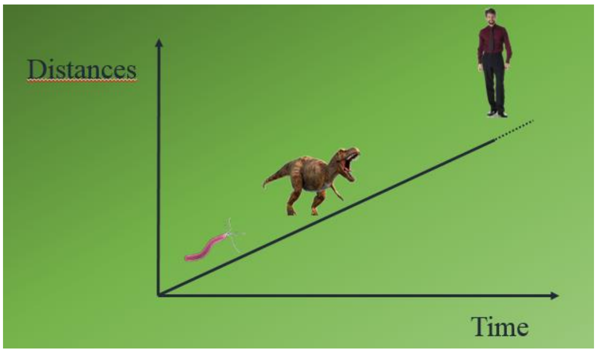
Putting these probabilities in a rate matrix we obtain this. From which Jukes and Cantor obtained the number K that is the estimated evolutionary distance between sequences

K = -3/4 ln [1 – (4/3)p]

	A	C	G	T
A	1-3α	α	α	α
C	α	1-3α	α	α
G	α	α	1-3α	α
T	α	α	α	1-3α

With «p» representing the observed differences between sequences while fractions ¾ and 4/3 indicate that the four nucleobases can be substituted in three different ways all equiprobable.
The union of all these hypothesis corresponds to a linear dependence of the distances between nucleobases based on time.

DNA AND EVOLUTION

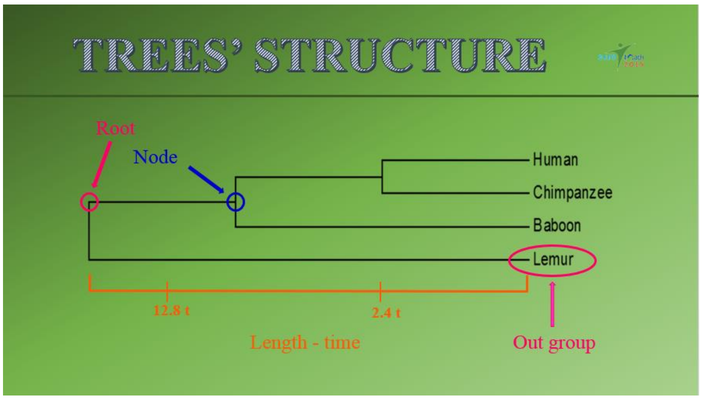


Even few percentage differences of nucleobases originate different individuals. In our procedure we measured these differences between two individuals by comparing the alignments of nucleobases of correspondent sections of DNA. The species we have considered in order to demonstrate this are four: humans, chimpanzees, baboons and lemurs.

```
Hu TAACTAGCAAATTTTCAGAGCTAGGGATAAAATAGGTAATTTTCAGAGCA
Ch TAACTAGCAAATTTTCAGAGCTAGAGCTAAAATAGGTAATTTTCAGAGCA
Ba TAACTAACAAATTATCAGAGCTACAGATAAAATAGGTAATATTAAGAGCA
Le AACAGAGTGAAACTCTGTCTCAAAAAAAAAAAAAGGAAGAGGACTTGGGA
   *   *   **         * *     **** *** *         * *
```

We used asterisks to underline the differences between nucleobases of the four primates.

PHYLOGENETIC TREES' STRUCTURE

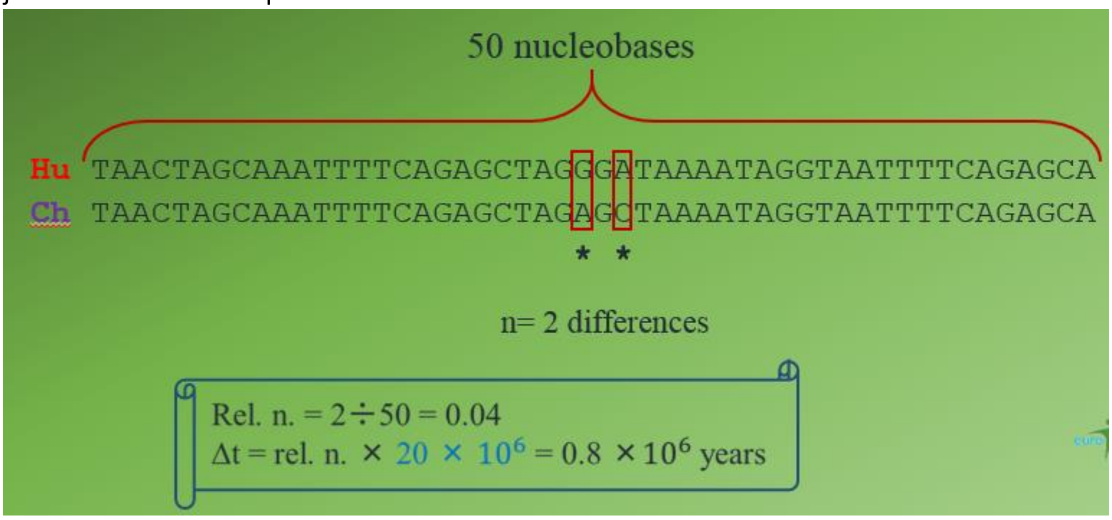


They are graphs that show the fundamental relationships of offspring between different organisms.

The length of the tree branch indicate that the change between nucleotides is directly proportional to the time: the longer is the branch, the longer is the time necessary for the birth of a new species.

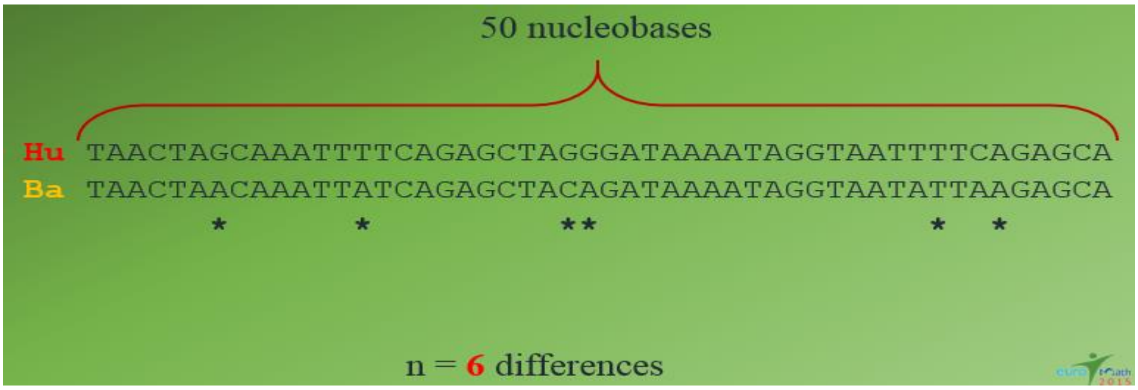
OUR CALCULATIONS WITH PAPER AND PEN

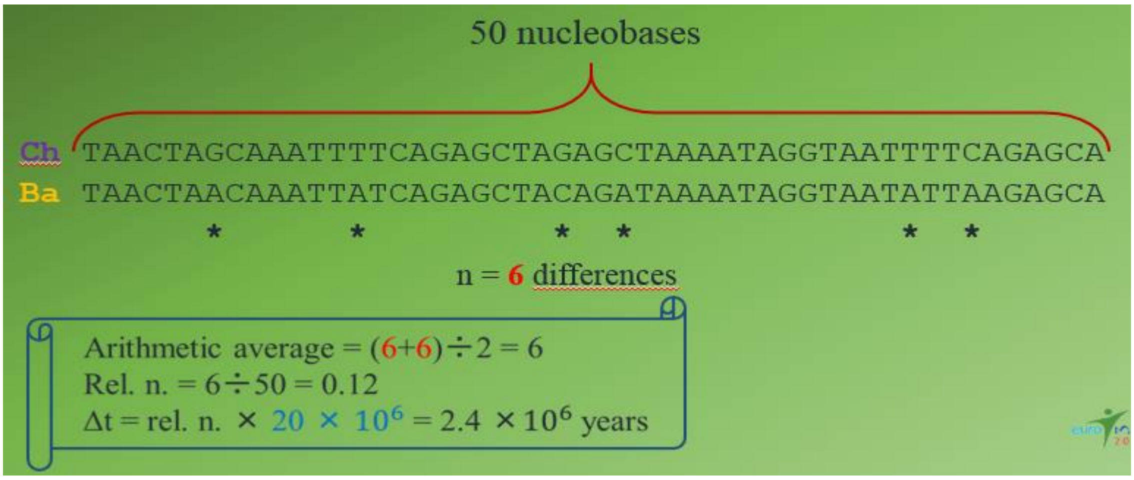
We have assumed that it would take 20 million years for every single nucleobase to change. Thus we have multiplied this number for the ratio between the number of differences of the species and the number of the considered nucleobases (rel.n). At the beginning we considered just humans and chimpanzees.



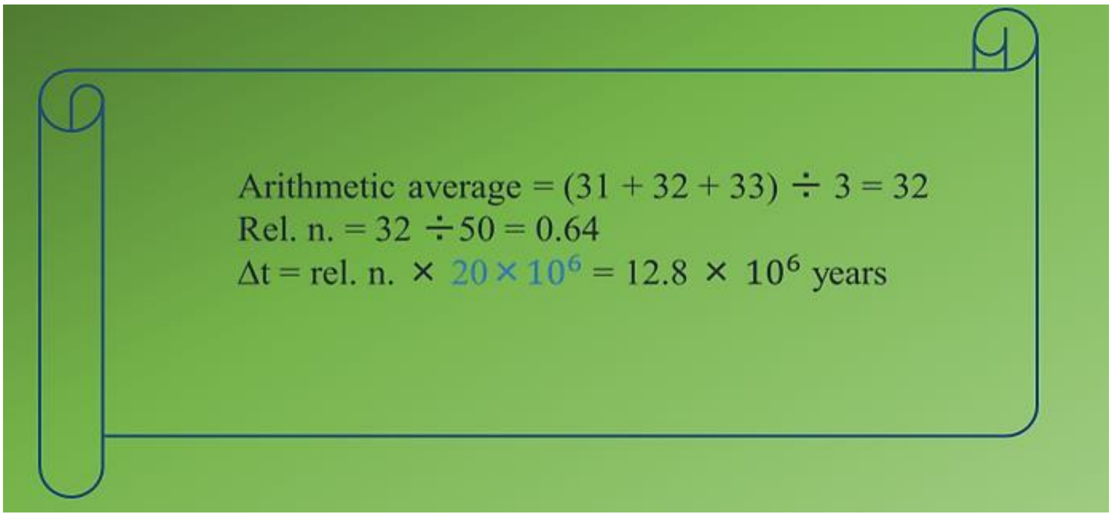
Then we added baboons considering the species in couples

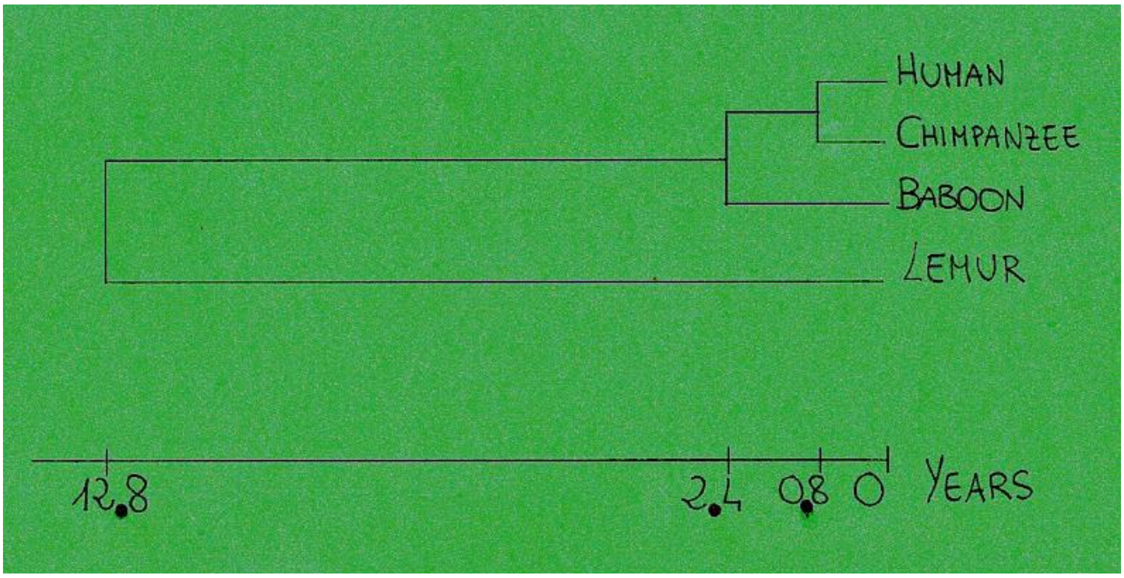
Finally we did the same also with lemurs. We compared each primate with this one, counted the number of differences, did the arithmetic average and calculated the relative number and





the time distance.





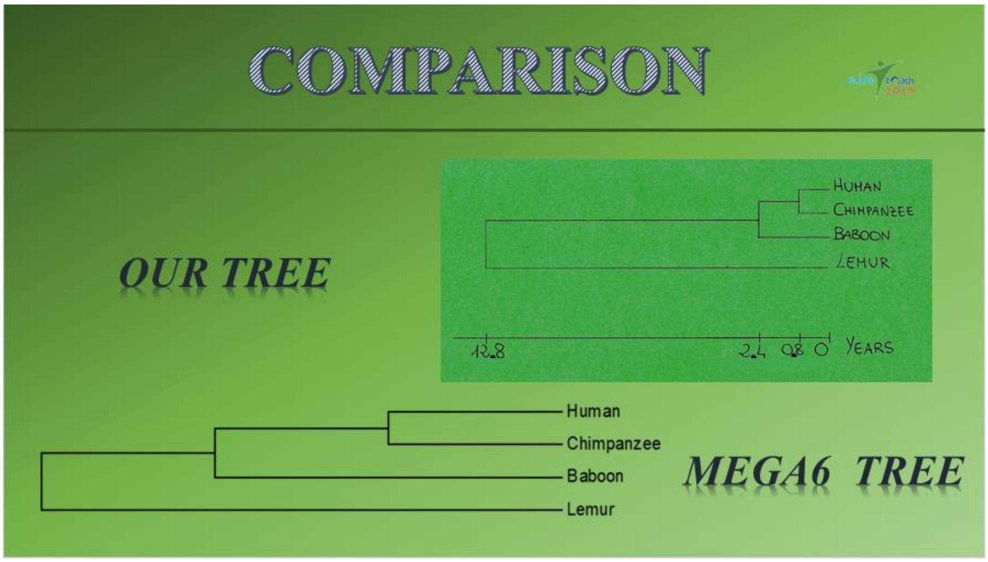
The result is our phylogenetic tree

COMPARISON WITH MEGA6

Megs6 is a program largely used by geneticists to model DNA evolution and we have filled in it the same alignments used in the previous calculation.

Unfortunately, Mega6 builds phylogenetic trees without any gaps calculation between species as we made.

In conclusion, our approach with paper and pen is exactly compatible with standardised procedures of calculation.



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THE MYSTERIES OF THE CIRCLE INVERSION

Jelena Družetić*, Antonia Mačukat, Hannah Villi Bigović***, Barbara Copić******
* jelena.druzetic@gmail.com,** amacukat@gmail.com, ***hannahvilli@gmail.com,
****barbara.copic21@gmail.com

ABSTRACT

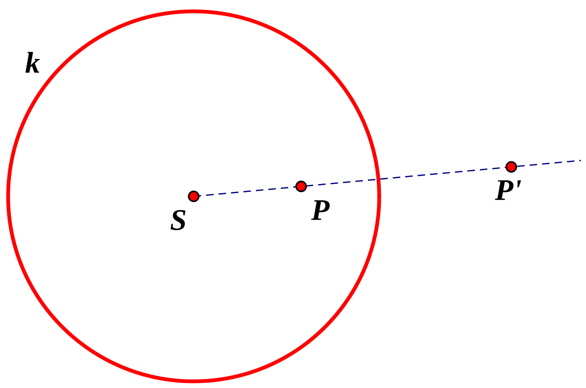
During the last school year we had been solving traditional geometrical problems from Japanese temples, which were also presented on the Euromath. We wanted to expand our horizons so we attended a museum exhibition called 'I Love Mathematics' (it is an international exhibition which was held in our town whose author is Georg Schierscher) and chose inverse geometry. It is the area of mathematics which has gained noticeable developments in the last two centuries and it is still active and popular area for researches. We were interested in different shapes which appear by using circle inversion. Furthermore, we started exploring relations and patterns inside and outside the circle. We found the ratios of the circles and objects inside them challenging and started thinking of a way to copy them outside. Moreover, we were fascinated by circle inversion of Sierpinski triangle in which the circle is situated in the middle "empty" triangle. We noticed that it is still a fractal. This challenge was accepted and we started to construct using the program of dynamic geometry ShetchPad. Our project assignment covers mathematical contents which are not part of regular school curriculum, and the aim is to awake curiosity for geometry in others.

INTRODUCTION

Circle inversion is, in a lot of ways, unexplored part of Euclidean Geometry. It has its connections in Hyperbolic Geometry, but we will only show you the Euclidean component of it.

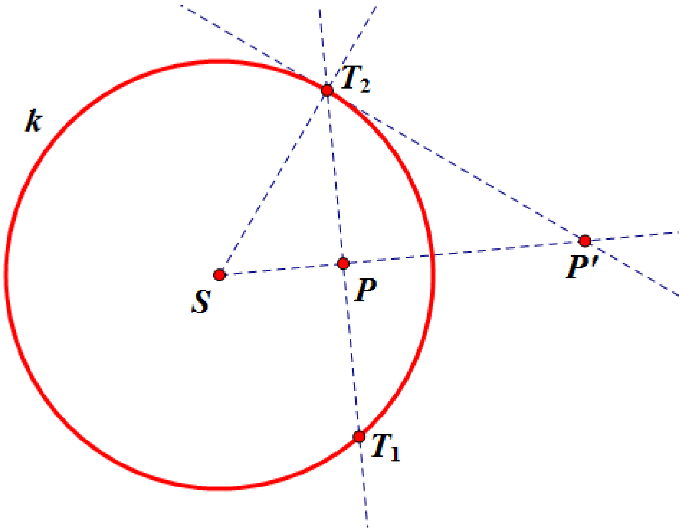
The invention of the transformation of inversion is sometimes credited to L. J. Magnus and sometimes to Jakob Steiner and Lord Kelvin. Regardless of who deserves credit for the invention of inversion, many mathematicians worked during the 1830s and 1840s to further develop the general theory.

DEFINITION



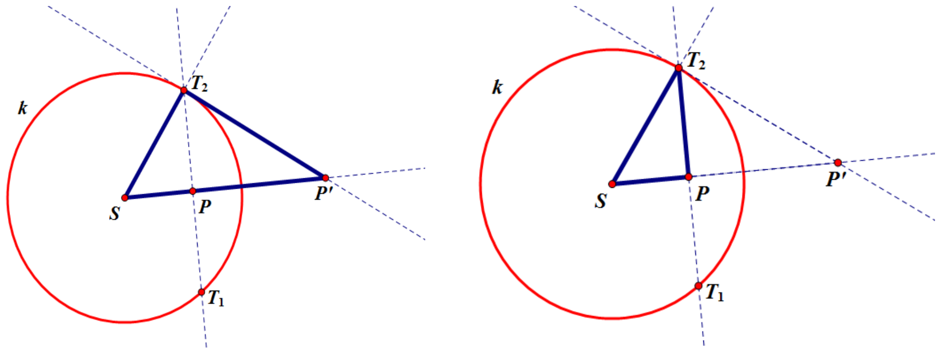
Firstly, we should explain the definition of the circle inversion. Let's call the circle of inversion circle k and the centre of circle k be S. If P is the point to be inverted, then its image, P', satisfies the given definition: |SP| * |SP'| = r2, where r is the radius of circle k.

How can you construct an inverse point? Begin with a circle k and point P inside the circle. Construct a line between the centre of circle k, which is point S, and P. Construct a line perpendicular to line SP through P. This line will intersect circle k at two points. Call these points T_1 and T_2. Draw lines connecting S with T_1 and S with T_2. Construct a line perpendicular to ST_1 through T_1 and a line perpendicular to ST_2 through T_2. The intersection of these lines is your inverse point, P'.



EXPLANATION

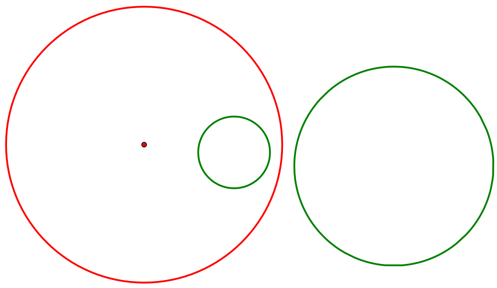
Angle SPT_2 is a right angle by construction. Angle ST_2P' is also a right angle by construction. Triangle SPT_2 is similar to triangle ST_2P' since they share angle T_2SP' and by AA similarity. We know that corresponding sides in similar triangles are proportional. Since $|ST_2| = r$, we have the proportion $|SP|/r = r/SP'$. Rearranging this equation gives $r^2 = |SP| * |SP'|$. This is the definition of inversion.



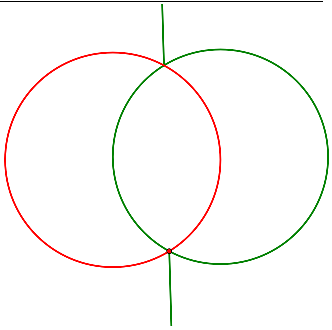
PROPERTIES OF CIRCLE INVERSION

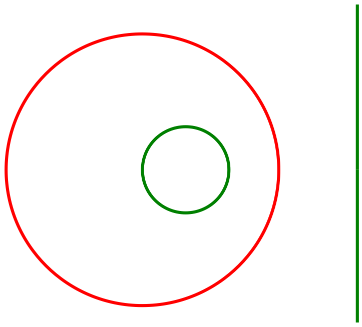
Circle inversion has many properties, but I am going to show just the ones that were most important to us and the basic ones.
One of the rules tells us that every point on the circle of inversion is fixed, which means that if P lies on circle k, then P = P’. Similarly, if P = P', then P must lie on circle k. If P is inside circle k, the P' is outside circle k. If P is outside circle k, the P' is inside circle k. Also, inversion of and inverted point makes the original point (P')' = P.

When under the transformation of inversion, lines are variant and circles invert into circles, but if we consider lines as circles of infinitive radii, then transformation of circle inversion is invariant.

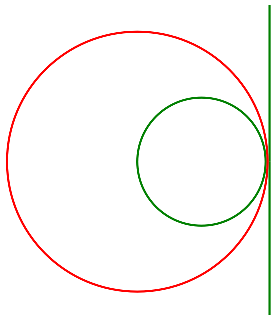


If a circle that we want to invert goes through the centre of the circle of inversion, its inversion will be a straight line. Here are some examples of it.

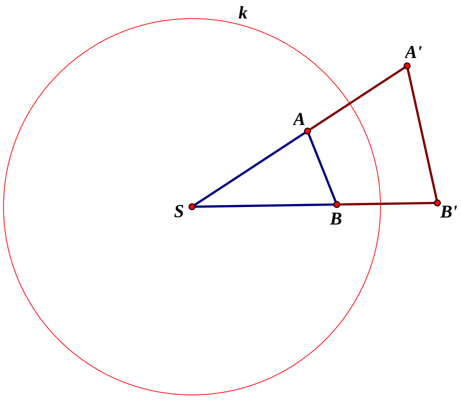




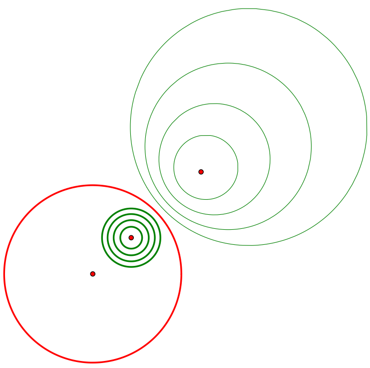
Circles of inversion are shown red and the circle that we inverted and its image is shown green.



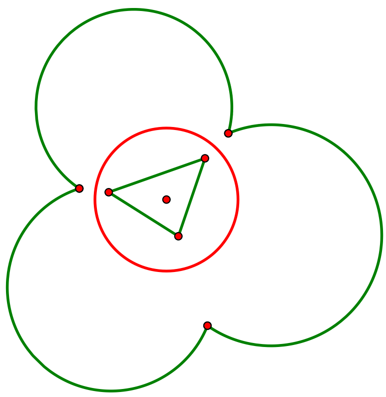
The measure of the angle between two intersecting objects is an invariant under the transformation of inversion. Using the main formula you can easily come across with the following equation: $A'B' = r^2 * \frac{AB}{SA*SB}$. This is the distance formula that relates the lengths of segments AB and A'B' where A' and B' are inversions of A and B with center S and radius r. We will use it for the next assignment which is called Ptolemy by Inversion.



Inversion with respect to a circle does not map the centre of the circle to the centre of its image.



As an example of the circle inversion, you can see here a triangle, which is inverted and when inside the circle of inversion makes arcs outside of the circle.

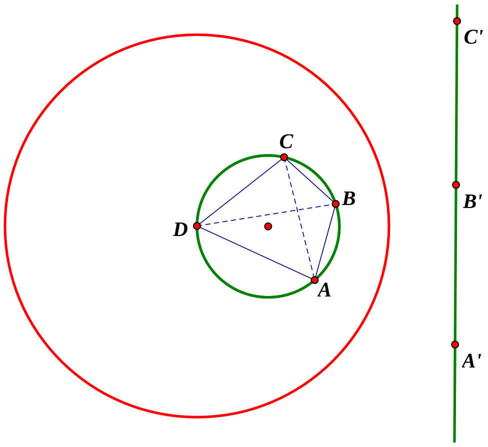


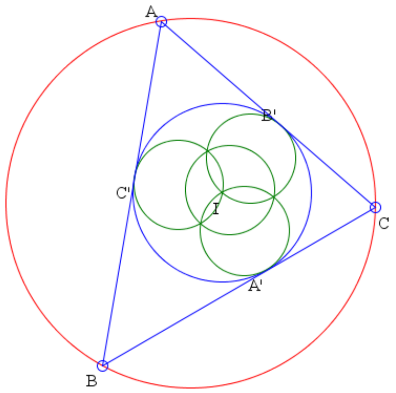
PROBLEMS

Ptolemy by Inversion

Let a convex quadrilateral ABCD be inscribed in a circle. Then the sum of the products of the two pairs of opposite sides equals the product of its two diagonals. In other words: AD·BC + AB·CD = AC·BD. We invert the whole configuration in the circle with centre D and (some) radius r.

By that inversion, the circumcircle of ABCD maps onto a straight line with images A', B', C' of A, B, C, respectively. For those images we do have A'B' + B'C' = A'C'. At this point we recollect the Angle Preservation Property of the inversion and its consequence - the distance formula that has been recently explained.



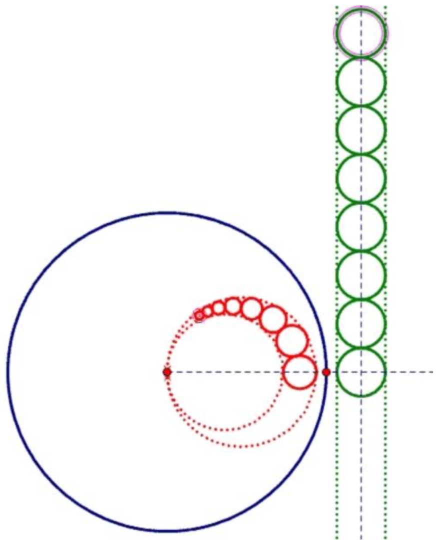


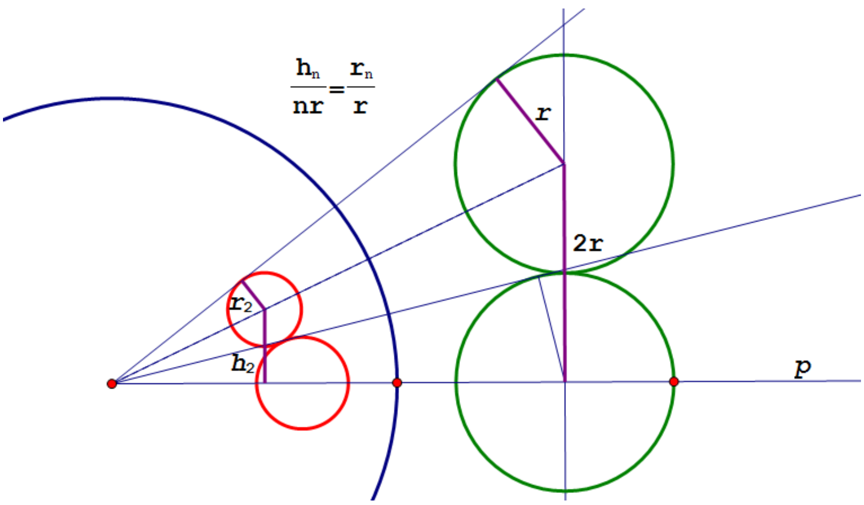
INVERSION IN THE INCIRCLE

One of the many interesting acknowledges that we came up with is the inversion in the incircle. It is said that the inverse images of the side lines of a triangle in its incircle are three circles of equal radii that concur at the incentre. The circle through their second points of intersection is none other than the inverse image of the circumcircle. It has the same radius.

CHAIN OF INSCRIBED CIRCLES

In this picture the circle of inversion is shown blue. Considering that circle, take a look at the inverted picture of a row of circles that are tangents to each other and inscribed to a pair of parallel lines. As it was told earlier, the image of parallel lines are circles that go through the center of the circle of inversion, and inside whom there is inscribed chain of circles. We can easily come to a result $h_n = 2n \cdot r_n$, where h_n is distance from the center of circle n from line p, a where r_n is her radius.





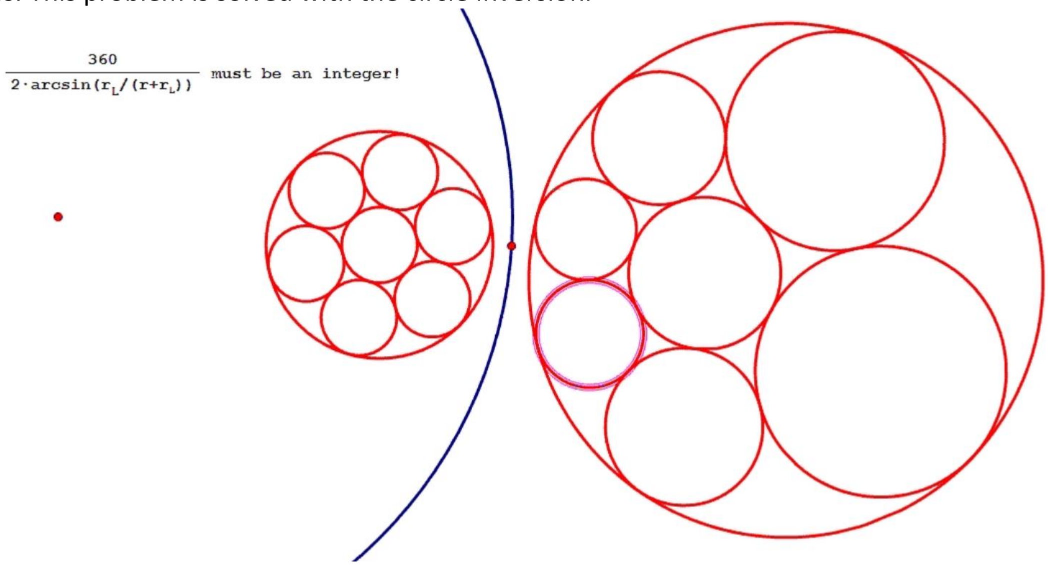
STEINER CHAIN

We were intrigued with work of one of the mathematicians that developed the general theory, Jakob Steiner. He was dealing with a problem that is now known as Steiner chain.
The question that Steiner chain of circles asks is can we make a configuration of circles, so that each one is touching two neighbouring ones and the center one.

In these two pictures we can see that they don't fulfil conditions,

but this picture here shows us that conditions are fulfilled.

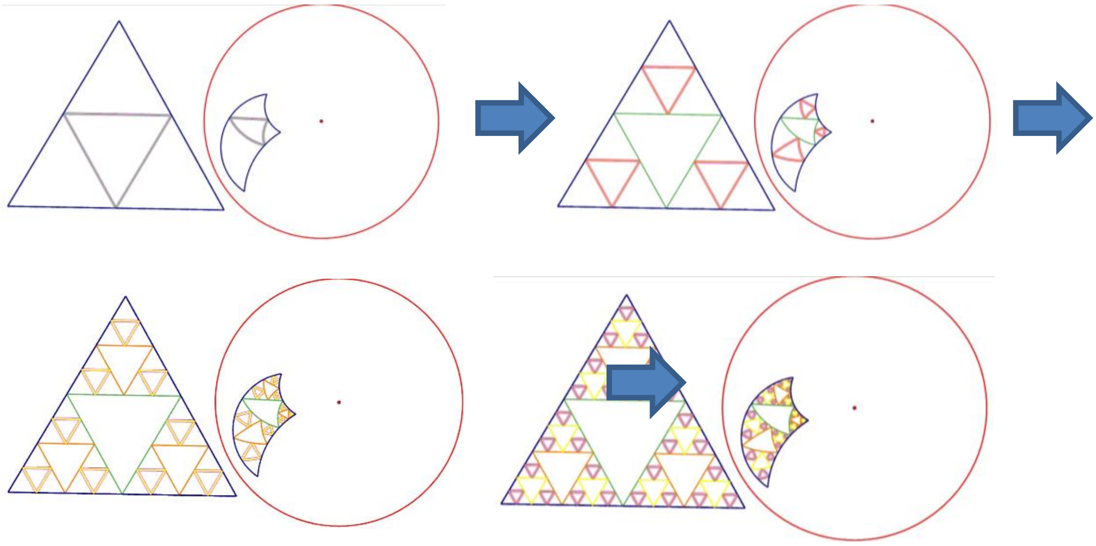
We can simply show conditions under which are possible to inscribe a chain of circle like those between two concentric circles. Much more complex question is if it's possible to inscribe a chain of circles as those inside two non-concentric circles. This problem is solved with the circle inversion.



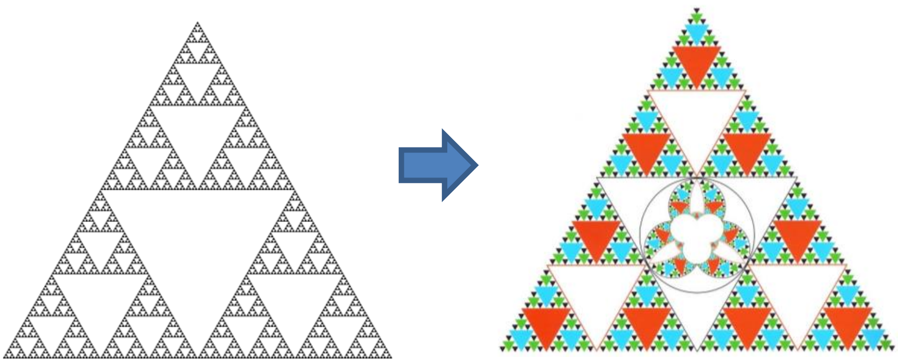
Sierpiński triangle

Sierpiński triangle is a fractal which is created like this:

It is also called the Sierpiński gasket or Sierpiński sieve. It was described by Sierpiński in 1915 and it appeared in Italian art from the 13th century. We wanted to invert three outer triangles into a circle that is inscribed to the center triangle. This pictures show us how to invert it.

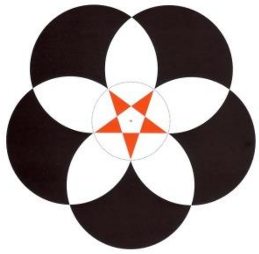


The wanted picture is shown here.



INTERESTING SHAPES

While we were exploring circle inversions, we came across with some interesting shapes by which we were amazed.
Inside the circle there are five isosceles triangles which bases are creating a pentagon. Points opposite of the bases are touching the circle. They invert into a flower.



If bases of isosceles triangles are tangent lines of the circle, then they are inverted into the circle.
Even the things from everyday life can show us examples of the circle inversions. This teapot, when situated in the center of the checked board, looked from above, gives us a picture of inverted board.

Here you can see Euromath logo also inverted.

Symmetry & Symmetry Breaking as Creative Catastrophe

Aristoklis Lagos
Athens College, Athens, Greece
lagos.aristoklis@gmail.com
Supervisor: Dr. Tefkros Michaelides, Mathematician

ABSTRACT

Symmetry, which derives from the Greek word "συμμετρία" (symmetria), refers to a sense of harmonious and beautiful proportion and balance. The opposite of symmetry is asymmetry. In the field of mathematics symmetry has a more precise definition.

This paper approaches symmetry from the following perspectives: a brief description of the historical roots and the emergence of the concept of symmetry that is at work in modern science (mathematics, physics, biology, chemistry) in arts as well as in social and political thought.

Every symmetrical structure reflects an equilibrium, a balance and a quasi-accepted situation in an existing structure. Symmetry can be exact, approximate or broken. The study of the symmetry breaking phenomenon is as serious as the study of symmetry. The creativity of symmetry and symmetry breaking consist a dynamic continual process with the name "creative catastrophe". The symmetry breaking in a structure is a generic process for the emergence of a new structure (morphi).

Furthermore, in this paper we are going to emphasize the general philosophical issues which are raising from the symmetry and the symmetry breaking. The symmetry exists in nature and in science as a concept. In human things symmetry does not exist and the symmetry breaking produces events with political and social consequences.

Additionally, we are going to attempt answering fundamental questions for symmetry like the following: Why does symmetry exist? Which are the meanings and the uses of symmetry? Is democracy the ideal political system where symmetry can flourish?

INTRODUCTION

Symmetry, and the opposite asymmetry, is the most fundamental concept in the human history and, meanly, in modern science. The symmetries are perfect either as empirical or as theoretical ones. While empirical symmetries relate situations, theoretical symmetries relate models of a theory we use to represent them. An empirical symmetry is perfect if and only if any two situations it relates share all intrinsic properties. Sometimes one can use a theory to explain an empirical symmetry by showing how it follows from a corresponding theoretical symmetry. The theory then reveals a perfect symmetry.

Symmetry is not alone, and symmetry breaking are twin sisters. Someone can evaluate the reality from the epistemological view of structuralism or the systems theory. So, symmetry in every structure or system reflects equilibrium, a balance and an accepted situation/condition in the existing structure/system. Symmetry can be exact, approximate, or broken. Exact means unconditionally valid; approximate means valid under certain conditions; broken can mean different things, depending on the object considered and its context.

Generally, the breaking of certain symmetry does not imply that no symmetry is present, but rather that the situation where this symmetry is broken is characterized by a lower/different symmetry than the original one. In group-theoretic terms, this means that the initial symmetry group is broken to one of its subgroups. It is therefore possible to describe symmetry breaking in terms of relations between transformation groups, in particular between a group (the unbroken symmetry group) and its subgroup(s).

Generally, the breaking of certain symmetry does not imply that no symmetry is present, but rather that the situation where this symmetry is broken is characterized by a lower symmetry than the original one. In group-theoretic terms, this means that the initial symmetry group is broken to one of its subgroups. It is therefore possible to describe symmetry breaking in terms of relations between transformation groups, in particular between a group (the unbroken symmetry group) and its subgroup(s).

The study of the “phenomenon” of symmetry and symmetry breaking has philosophical perspectives and is very important for the knowledge of the nature, science, or of the “human things” (society and politics). Symmetry and symmetry breaking consist a dynamic continual process with the name “creative catastrophe”. This creative catastrophe is a generic process, a morphogenesis for emergence/development of a new structure (morphi).

In this paper, firstly, we describe the historical roots and the emergence of the concept of symmetry that is at work in modern science. After, we present the existence of symmetry and symmetry breaking in the fields of nature, mathematics, modern physics, biology/chemistry; in arts; in social and political interactions. Furthermore, we are going to emphasize the general philosophical issues and perspectives which are rising from the symmetry and symmetry breaking. Additionally, we are going to answer fundamental ontological questions concerning symmetry. In human things symmetry does not exist and the symmetry breaking produces events with political and social consequences. Also we are going to estimate if democracy is the ideal political system into which symmetry can flourish.

Finally, we elevate the symmetry as the fundamental imagination of human society which presents an energetic and decisive challenge to scientists, philosophers and citizens.

1. SETTING THE PROBLEM: THE HISTORICAL CONTEXT OF SYMMETRY

Like any other scientific concept, symmetry has a history. The concept of symmetry, as it is currently applied in many scientific domains, is entirely different from that was meant by the term symmetry in ancient and medieval times up to the early modern period. The scientific concept of symmetry, as we know it today, is a 19th century concept.

Symmetry (Greek: συμμετρία, symmetria) had one basic meaning in Greek antiquity: proportionality. Its usage can be distinguished be the following contexts:

- In a mathematical context it means that two quantities share a common measure (i.e. they are commensurable), and
- In an evaluative context (e.g. appraising beauty) it means well proportioned.

Both contexts constitute two different backgrounds for two different paths in the evolution of the concept. The coherence of these two trajectories corresponds to two different distinct senses of the concept of symmetry: (1) a relation between two entities, and (2) a property of a unified whole.

Symmetry in its current scientific usage refers either to a mathematico-logical relation or to an intrinsic property of a mathematical entity which under certain classes of transformations, such as rotation, reflection, inversion, or other abstract operations, leaves something unchanged-invariant. When an invariant property is maintained, it is the subject of Group Theory. A mathematical theory which explores systematizes and formalizes features that are preserved under the transformation. In fact, the aesthetic sense of symmetry can be described mathematically, but the essential point is that in modern scientific usage symmetry is mathematical element with no aesthetic component.

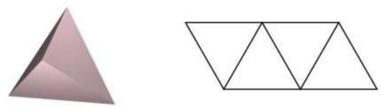
2. THE MATHEMATICAL PATH OF SYMMETRY

Plato and his Five Perfect Solids

Plato was a Greek philosopher who lived from 427 BC to 347 BC. He was keenly interested in solid geometry and its place in the workings of our universe. To Plato, symmetry was a fundamental property of the universe, and geometry was a tool to know symmetry.

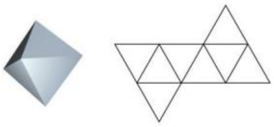
Plato is noted for his observations of the set of five regular solids that bear his name. There are polyhedra with symmetrical faces, edges and vertices – perfect solids. He spoke of an atomic universe comprised of four elements: fire, air, earth and water. Four of his solids formed these elements, and the fifth formed the universe itself. And so Plato taught that ideal forms spawn everything in the universe. He thought of nature as a complex system based on diverse instantiations of these ideal forms. The more perfect the form, the more closely it approached whatever truth may lie at the heart of the universe. This seems so highly metaphysical by today's standards as to be useless, but modern scientific abstraction is heading back in this direction. Only five polyhedrons in the observable universe can be perfectly symmetric, so here is a brief overview of all five of Plato's most perfect forms.

Tetrahedron



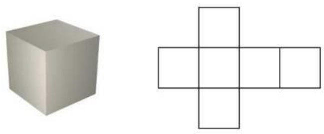
This solid has four triangular faces, four vertices, and six edges. It is dual to itself. The acuteness of its angles led Plato to name it fire.

Octahedron



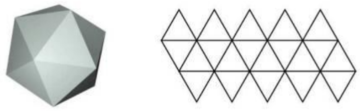
This solid has eight triangular faces, six vertices and twelve edges. It is the dual of the cube. Air is the name given to the octahedron, because it was seen as an intermediate between fire and water.

<u>Cube</u>



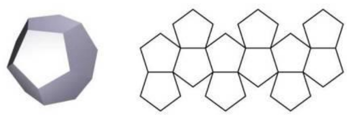
This solid has six square faces, eight vertices and twelve edges. It is the dual of the octahedron. The stability of the cube led Plato to associate it with the element earth.

<u>Icosahedron</u>



This solid has twenty triangular faces, twelve vertices and thirty edges. It is the dual of the dodecahedron. Plato called the icosahedron water.

Dodecahedron



This solid has twelve pentagonal faces, twenty vertices and thirty edges. It is the dual of the icosahedron. This is the most mysterious and powerful of the five regular solids. It embodies the other four; Plato therefore said that the dodecahedron is the cosmos. He sensed that it was used by God to embroider the heavens.

Digression:

a) Plato took the bar from Pythagoras concerning geometry.
b) Plato's perfect solids have their origin from the symmetries of regular triangles and pentagons. The equilateral triangle has six symmetries. The perfect pentagon has five symmetries. A circle or a sphere has infinite symmetries.
c) In the 20th century, attempts to link Platonic Solids to the physical world were expended to the electron shell model in chemistry by Robert Moon in a theory known as the "Moon Model".

Jean d' Alembert (1751-1765) the Encyclopedia
This word (asymmetrie) designates in mathematics what one ordinarily understands now by incommensurability. There is incommensurability between two quantities when they do not have any common measure, as for example the side of a square and its diagonal; in numbers –like the square root of two– are also incommensurable with rational numbers. On the Eve of French Revolution symmetry and asymmetry were no longer current terms in mathematical usage.

Johannes Kepler
The German astronomer attempted to relate the five extraterrestrial planets known at that time to the Platonic Solids (model: the harmony of the spheres). So the concept of harmony was identified with the concept of symmetry.

Evariste Galois
O Evariste Galois (25 October 1811 – 31 May 1832) was a French mathematician born in Bourg-la-Reine. While still in his teens, he was able to determine a necessary and sufficient condition for a polynomial to be solvable by radicals, thereby solving a 350 years-standing problem. His work laid the foundations for Galois Theory and Group Theory, two major branches of abstract algebra, and the subfield of Galois connections. He died at age 20 from wounds suffered in a duel. The group theory is modern mathematic view point for the symmetry as a result of transformations groups.

Eugene Wigner (1902-1995)
The doyen of the application of symmetry in physics, received the Nobel Prize in physics in 1963 "for his contributions for discovering the fundamental symmetry principles". The hierarchy of knowledge of the world progresses, according to Wigner, from events to law of nature, and from laws to symmetry or invariance principles.

Hermann Weyl (1855-1955)
In his masterful work "Symmetry", connects three distinct domains with the concept of symmetry: (1) material artifacts, (2) natural phenomena, and (3) physical theories. Weyl proved that Group Theory is the underlying mathematical structure for symmetry in all domains.

Felix Klein – Erlangen Programme
The Erlangen program is a method of characterizing geometries based on group theory and projective geometry. The way the multiple languages of geometry then came back together could be explained by the way subgroups of a symmetry group related to each other. The long-term effects of the Erlangen program can be seen all over pure mathematics; and the idea of transformations and of synthesis using groups of symmetry is of course now standard too in physics. Felix Klein said: *"Before Erlangen programme Symmetry was appeared from the geometry, but, now the Geometries are the consequence of Symmetry".*

3. THE AESTHETIC PATH OF SYMMETRY

The noun symmetria and its adjective symmetros were common words in Greek and used in many different contexts (in addition to mathematics), with a variety of meanings relating to the notion of "properly proportioned". These meanings are given in the standard Greek lexicon as: "due proportion", a characteristic of beauty and goodness; suitability; convenient in size; moderate in size; fitting or appropriate.

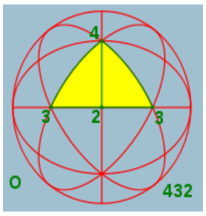
Plato: Beauty as Symmetry
Towards the end of "Timeous" Plato discusses the physiology of animals with an emphasis on the human body. He focuses on the relation between the soul and body. He argues that, in order for a living creature to be beautiful, it must be symmetrical – that is well proportioned.

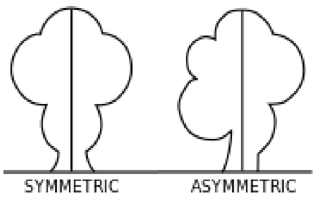
In this approach, beauty is always an aspect of good and it implies well proportioned, but Plato does not specify the criteria. Plato precedes a more pressing question: How could one establish proportions between soul and body? This aspect of symmetry involves an aesthetic evaluation of a specific relation, a specific proportion, but a precise definition in terms of numbers is not given.
Moreover, it is clear from the context that ametron is a synonym for assymetria.

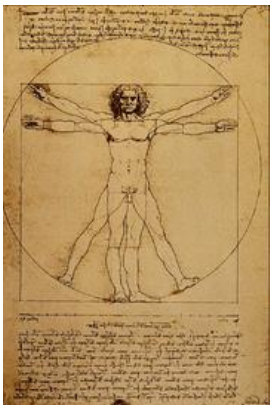
Maurits Cornelis Escher: the innovation, the creative catastrophe in the Arts
Maurits Cornelis Escher created unique and fascinating works of art that explore and exhibit a wide range of mathematical ideas.
As his work developed he drew great inspiration from the mathematical ideas he read about, often working directly from structures in plane and projective geometry, and eventually capturing the essence of non-Euclidean geometries. He was also fascinated with paradox and "impossible" figures, and used an idea of Roger Penrose's to develop many intriguing works of art. Thus, for the student of mathematics, Escher's work encompasses two broad areas: the geometry of space, and what we may call the logic of space. The Escher's works produce an "imaginary/pictorial philosophy". (Tefkros Michaelides, Dr. Mathematician).

4. SYMMETRY AND SYMMETRY BREAKING IN PARADIGMS/FIELDS

Some pictures:

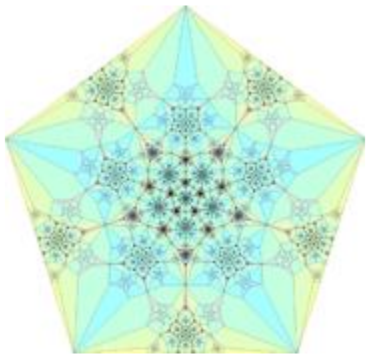






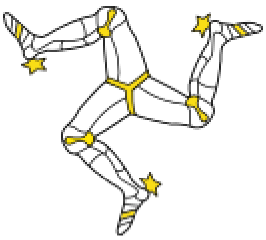
Sphere symmetrical group representing an octahedral rotational symmetry. The yellow region shows the fundamental domain.
Leonardo da Vinci's 'Vitruvian Man' (ca. 1487) is often used as a representation of symmetry in the human body and, by extension, the natural universe.

A fractal-like shape that has reflectional symmetry, rotational symmetry and self-similarity, three forms of symmetry. This shape is obtained by a finite subdivision rule.



Symmetric arcades of a portico in the Great Mosque of Kairouan also called the Mosque of Uqba, in Tunisia.
Symmetry (from Greek συμμετρία "symmetria") agreement in dimensions, due proportion, in everyday language refers to a sense of harmonious and beautiful proportion and balance. In mathematics, "symmetry" has a more precise definition, that an object is invariant to a transformation, such as reflection but including other transforms too. Although these two meanings of "symmetry" can sometimes be told apart, they are related, so they are here discussed together.
This paper describes symmetry from three perspectives: in mathematics, including geometry, the most familiar type of symmetry for many people; in science and nature; and in the arts; in social and political interactions / relations.
The opposite of symmetry is asymmetry.

Symmetry in Geometry



The triskelion has 3-fold rotational symmetry.
A geometric shape or object is symmetric if it can be divided into two or more identical pieces that are arranged in an organized fashion. This means that an object is symmetric if there is a transformation that moves individual pieces of the object but doesn't change the overall shape. The type of symmetry is determined by the way the pieces are organized, or by the type of transformation:
An object has reflectional symmetry if there is a line of symmetry going through it which divides it into two pieces which are mirror images of each other. An object has rotational symmetry if the object can be rotated about a fixed point without changing the overall shape. An object has translational symmetry if it can be translated without changing its overall shape. An object has helical symmetry if it can be simultaneously translated and rotated in three-dimensional space along a line known as a screw axis. An object has scale symmetry if it does not change shape when it is expanded or contracted. Fractals also exhibit a form of scale symmetry, where small portions of the fractal are similar in shape to large portions.

Symmetry Breaking in Geometry
Quadrature circle; doubling cube; the Erlangen Programme of Felix Klein (which generalized the Euclidean and non-Euclidean geometries).

Symmetry in Mathematics
Generalizing from geometrical symmetry in the previous section, we say that a mathematical object is symmetric with respect to a given mathematical operation, if, when applied to the object, this operation preserves some property of the object. The set of operations that preserve a given property of the object form a group.
In general, every kind of structure in mathematics will have its own kind of symmetry. Examples include even and odd functions in calculus; the symmetric group in abstract algebra; symmetric matrices in linear algebra; and the Galois group in Galois Theory.

Symmetry Breaking in Maths
The Higgs mechanism of mass generation is the main ingredient in the contemporary Standard Model and its various generalizations. However, there is no comprehensive theory of spontaneous symmetry breaking. We summarize the relevant mathematical results characterizing spontaneous symmetry breaking phenomena in algebraic quantum theory, axiomatic quantum field theory, group theory, and classical gauge theory.

Symmetry in Physics

The modern scientific definition of symmetry is that of invariance under transformations.

Note, however, that symmetry remains linked to beauty (regularity) and unity: by means of the symmetry transformations, distinct (but “equal” or, more generally, “equivalent”) elements are related to each other and to the whole, thus forming a regular “unity”. The way in which the regularity of the whole emerges is dictated by the nature of the specified transformation group. Summing up, a unity of different and equal elements is always associated with symmetry, in its ancient or modern sense; the way in which this unity is realized, on the one hand, and how the equal and different elements are chosen, on the other hand, determines the resulting symmetry and in what exactly it consists.

The definition of symmetry as “invariance under a specified group of transformations” allowed the concept to be applied much more widely, not only to spatial figures but also to abstract objects such as mathematical expressions — in particular, expressions of physical relevance such as dynamical equations. Moreover, the technical apparatus of group theory could then be transferred and used to great advantage within physical theories.

There are two main ways of using symmetry. First, we may attribute specific symmetry properties to phenomena or to laws (symmetry principles). It is the application with respect to laws, rather than to objects or phenomena, that has become central to modern physics, as we will see. Second, we may derive specific consequences with regard to particular physical situations or phenomena on the basis of their symmetry properties (symmetry arguments).

Important symmetries in physics include continuous symmetries and discrete symmetries of space time; internal symmetries of particles; and supersymmetry of physical theories

Symmetry Breaking in Physics

Generally, the breaking of certain symmetry does not imply that no symmetry is present, but rather that the situation where this symmetry is broken is characterized by a lower symmetry than the original one. In group-theoretic terms, this means that the initial symmetry group is broken to one of its subgroups. It is therefore possible to describe symmetry breaking in terms of relations between transformation groups, in particular between a group (the unbroken symmetry group) and its subgroup(s). Starting from this point of view a general theory of symmetry breaking can be developed by tackling such questions as “which subgroups can occur?”, “when does a given subgroup occur?”

There are two different types of symmetry breaking of the laws: “explicit” and “spontaneous”, the case of spontaneous symmetry breaking being the more interesting from a physical as well as a philosophical point of view.

Explicit symmetry breaking indicates a situation where the dynamical equations are not manifestly invariant under the symmetry group considered.

Spontaneous symmetry breaking (SSB) occurs in situations where, given symmetry of the equations of motion, solutions exist which are not invariant under the action of this symmetry without any explicit asymmetric input.

Fundamental Questions and Philosophical Perspectives on the Modern Scientific Concept

The notion of symmetry and the symmetry breaking has demonstrated a fruitfulness and range of applicability.

But fundamental questions about the symmetry principles in science remain. Do symmetry principles convey empirical information about the world, or are they in some sense trivially true? Are symmetry considerations merely methodological sound procedures, or do they correspond to the structure of the world? Can the validity of symmetry principles be established a priori by logical means, or must it be learned by experience?

Much of the recent philosophical literature on symmetries in physics discusses specific symmetries.

To conclude:
The symmetries are perfect symmetries as empirical ones or theoretical symmetries as epistemological models: in physics offer many interpretational possibilities, and how to understand the status and significance of physical symmetries clearly presents a challenge to both physicists and philosophers.

Order vs. Disorder: The Symmetry of Chaos
Symmetry is understood as corresponding to a highly ordered state which requires effort to achieve and maintain. Departures from symmetry are understood to correspond to disorder and lack of structure. The concept of entropy and the thermodynamic arrow of time suggest the idea that systems in nature tend to decay from order to disorder. On a reading inspired by algorithmic information theory, symmetrical states of affairs correspond with low information, while disordered outcomes have higher information content. Beauty is associated with symmetry in classical theories of art, but many modern onlookers would associate perfect symmetries with lifelessness and lack of aesthetic value. What do these conceptual relations tell us about the relations of order and disorder?

Symmetry in Biology
Symmetry plays important roles in biological science too. The symmetry of plant morphology is regarded as a leading principle in ontogenesis. The bilateral symmetries of animals, and their potential assessment by potential mates, have been revealed as an important factor in sexual selection.
Bilateral animals, including humans, are more or less symmetric with respect to the sagittal plane which divides the body into left and right halves. Animals that move in one direction necessarily have upper and lower sides, head and tail ends, and therefore a left and a right. The head becomes specialized with a mouth and sense organs, and the body becomes bilaterally symmetric for the purpose of movement, with symmetrical pairs of muscles and skeletal elements, though internal organs often remain asymmetric.
Plants and sessile (attached) animals such as sea anemones often have radial or rotational symmetry, which suits them because food or threats may arrive from any direction. Five fold symmetry is found in the echinoderms, the group that includes starfish, sea urchins, and sea lilies.
The human body has reflectional symmetry but the internal organs appear a symmetry breaking in order to satisfy the functions.

Symmetry and Symmetry Breaking in Morphogenesis
The concepts of symmetry and symmetry breaking play an important role in accounts of morphogenesis in at least two widely differing contexts: cosmogenesis in cosmology and ontogeny in development biology. In both contests, a symmetrical early state results in an asymmetrical later state. What is the outcome structure due to the symmetry breaking? Can any systematic principles relating symmetry breaking to the evolution of structure be established? How far the outcome lies, due to symmetry breaking, from the initial symmetry?

Symmetry Breaking in the Early Universe
Our current understanding of particle physics and cosmology implies that the early Universe probably went through a series of symmetry breaking phase transitions as it cooled down and expanded to become what we know today.
From the theoretical point of view, the last few years have seen great progress in our understanding of the creation of topological defects in a symmetry breaking phase transition, largely through the interaction between cosmologists and condensed matter physicists in a beautiful example of interdisciplinary collaboration.

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Symmetry in Chemistry

Symmetry is important to chemistry because it undergirds essentially all specific interactions between molecules in nature (i.e., via the interaction of natural and human-made chiral molecules with inherently chiral biological systems). The control of the symmetry of molecules produced in modern chemical synthesis contributes to the ability of scientists to offer therapeutic interventions with minimal side effects. A rigorous understanding of symmetry explains fundamental observations in quantum chemistry, and in the applied areas of spectroscopy and crystallography. The theory and application of symmetry to these areas of physical science draws heavily on the mathematical area of group theory.

Symmetry Breaking in Chemistry

Certain molecules are chiral; that is, they cannot be superposed upon their mirror image. Chemically identical molecules with different chirality are called enantiomers; this difference in orientation can lead to different properties in the way they react with biological systems.

Symmetry in Architecture

The ceiling of Lotfollah mosque, Isfahan, Iran has 8-fold symmetries.

The Taj Mahal has bilateral symmetry.

Symmetry finds its ways into architecture at every scale, from the overall external views of buildings such as Gothic cathedrals and The White House, through the layout of the individual floor plans, and down to the design of individual building elements such as tile mosaics. Islamic buildings such as the Taj Mahal and the Lotfollah mosque make elaborate use of symmetry both in their structure and in their ornamentation. Moorish buildings like the Alhambra are ornamented with complex patterns made using translational and reflection symmetries as well as rotations. It has been said that only bad architects rely on a "symmetrical layout of blocks, masses and structures";

Symmetry Breaking in Architecture: Asymmetry as Innovation

Pre-modern architectural styles tended to place an emphasis on symmetry, except where extreme site conditions or historical developments lead away from this classical ideal. To the contrary, modernist and postmodern architects became much more free to use asymmetry as a design element.

While most bridges employ a symmetrical form due to intrinsic simplicities of design, analysis and fabrication and economical use of materials, a number of modern bridges have deliberately departed from this, either in response to site-specific considerations or to create a dramatic design statement.

Some asymmetrical structures

Eastern span replacement of the San Francisco – Oakland Bay Bridge

Puente de la Mujer

Auditorio de Tenerife

Symmetry in aesthetics

Symmetry in aesthetics shows a physical attractiveness.

The relationship of symmetry to aesthetics is complex. Humans' find bilateral symmetry in faces physically attractive; it indicates health and genetic fitness. Opposed to this is the tendency for excessive symmetry to be perceived as boring or uninteresting. People prefer shapes that have some symmetry, but enough complexity to make them interesting.

Symmetry Breaking in Aesthetics

Many modern onlookers would associate perfect symmetries with lifelessness and lack of aesthetic value.

Symmetry and Symmetry Breaking in the Arts

Maurich Cornelis Escher did not have mathematical training—his understanding of mathematics was largely visual and intuitive. Escher's work had a strong mathematical component, and more than a few of the worlds which he drew were built around impossible objects such as the "Penrose triangle" and the "Penrose stairs". Many of Escher's works employed repeated tilings called tessellations. Escher's artwork is especially well liked by mathematicians and scientists, who enjoy his use of polyhedral and geometric distortions. Escher's works produce an "imaginary philosophy" (Tefkros Michaelides), and an innovative interpretation of the symmetry and symmetry breaking concept.

5. ONTOLOGICAL AND PHILOSOPHICAL PERSPECTIVES FOR SYMMETRY

From where are the roots / origins if symmetry? There are only perfect symmetries, either as empirical symmetries i.e. in nature or scientific symmetries.

Which are the meanings and the important usage of symmetry? The main meaning of symmetry is the steady balance in a structure / system. Symmetry usages' is to accepting stability and order in a structure / system.
The symmetries are perfect symmetries as empirical ones or theoretical symmetries as epistemological models: in physics offer many interpretational possibilities, and how to understand the status and significance of physical symmetries clearly presents a challenge to both physicists and philosophers.
Symmetry and symmetry breaking are the fundamental concepts for a holistic evaluation of the reality.

6. SYMMETRY AND SYMMETRY BREAKING IN HUMAN THINGS, SOCIAL AND POLITICAL INTERACTIONS

Symmetry and symmetry breaking compose the Janus two faces of the unique phenomenon under the name Morphogenesis. This phenomenon produces the system development in its several situations or phases. Every system phase has its symmetry and the system breaking drives the system in other symmetry. Rene Thom (1923-2002), a French mathematician and Philosopher, a pioneer of Catastrophe Theory, has studied the morphogenesis phenomenon concerning development phases and the symmetries. (Rene Thom, *"The morphogenesis mathematical patterns", 19*75).
Among all the conditions which regulate a balance situation in a system the symmetry of conditions is reflected on the balance situation.
Symmetry, on one reading, is understood as corresponding to an equal development whereas unequal development is understood to correspond asymmetry and disorder of the society. Social inequalities constitute a symmetry breaking and threaten the stability of society and democracy. Sustainable development steadily produces the high symmetry in society.
The bipolar political system had symmetry in the world politics while today the multipolar system is a result of symmetry breaking and produces instability and disorder because it has lower symmetry.
Cyprus before the debt crisis was a democratic society with symmetry harmony. Debt crisis was a symmetry breaking which drove society and democracy in instability and citizens in disappointment.
Debt crisis in Greece is a symmetry breaking which threats the existing economic political and social system.
Climate change is a symmetry breaking into the holistic function of Gaia/Earth and it has be done due to men with their activities destroyed the existing symmetry of the Earth.
Guernica, an excellent Picasso work, has not any symmetry. Asymmetry, disorder and chaos are the main features. The lack of any symmetry means the absolute catastrophe.
Dimitris Nanopoulos, the famous cosmologist and physician, has developed theories for the existence many universes which are equivalent and similar with our universe. This scientific constitute a group of hyper symmetries which are not comparable with the known symmetries.
Symmetry exists in nature as event and in science as theory. In human things symmetry does not exist while a symmetry breaking is a creative catastrophe and produces events with political, social and cultural impacts. Human/Citizens could target and win a definite symmetry or symmetry breaking only in means of Politics.
People observe the symmetrical nature, often including asymmetrical balance, of social interactions in a variety of contexts. These include assessments of reciprocity, empathy, apology, dialog, respect, justice, and revenge. Symmetrical interactions send the message "we are all equal citizens" while asymmetrical interactions send the message "I am special; better than you." Peer/equal relationships are based on symmetry, power relationships are based

on asymmetry. Small economic inequalities drive society in order with symmetry. While strong inequalities produce a symmetry breaking which is the beginning for the social collapse?

In Ancient Athens Democracy was born. That democracy born Symmetry (Greek: συμμετρία) and Symmetry breaking due to continual laying down by the citizens. Symmetry was an innovative concept and value in politics, in society, in citizen interactions, in aesthetics, in architecture and in philosophy generally.

European Union is a political society which wants the symmetry and harmony in development among the member states. Instead of them, there is a development gap among the European north and south; symmetry does not exist a symmetry breaking is emerging which will produce a new more justice European symmetry.

Democracy with freedom and justice for equal but not similar citizens is the suitable environment for harmony and symmetry.

Democracy is the ideal/best political system where symmetry and symmetry breaking can creatively flourish.

7. CONCLUSION

Symmetry and symmetry breaking is the fundamental concept for the modern science.

The creativity of symmetry and symmetry breaking is a Morphogenesis – as a creative catastrophe - which continually produces new structures (*morphi*) in nature, in science and in society.

Symmetry exists as empirical symmetry and as scientific symmetry. In human things symmetry does not exist and the symmetry breaking produces events with political and social consequences.

Symmetry in science offers many interpretational possibilities how understand the state and significance of symmetries. On the other hand, symmetry is the fundamental imagination of human society. Symmetry and symmetry breaking, as a holistic phenomenon of reality, presents a challenge to scientists, philosophers and citizens.

Democracy –with freedom and justice among equal citizens– is the ideal/best political system where symmetry and symmetry breaking can creatively flourish.

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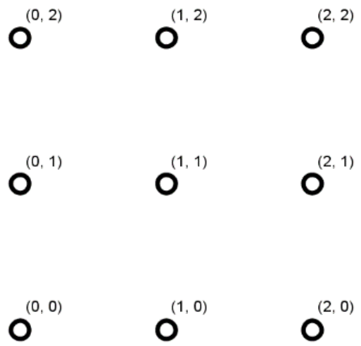
VARIOUS FIGURES ON A FINITE PLANE

Andrzej Szablewski, Jakub Mazur, Jakub Niewiarowski, Radosław Peszkowski, Mateusz Janus
Gimnazjum im. Jana Matejki w Zabierzowie, ul. Kolejowa 15, 32-084 Zabierzów, Poland

ABSTRACT

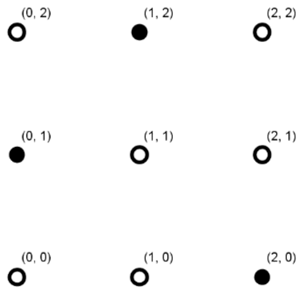
In the present note we investigate a plane which consists of 9 points. This is the plane defined over the field of 3 elements. We show, that already nine points are enough in order to study some nice geometry.

The geometry of the 9 points is the finite plane which consists of only nine points.



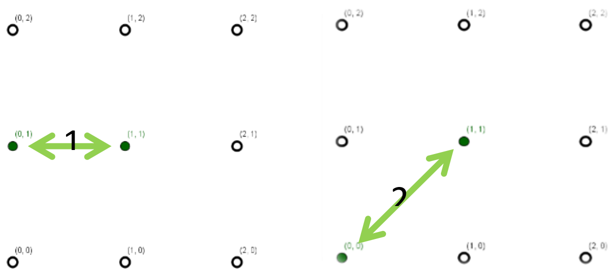
9 points plane

There are many figures which can be defined. We should start from the simple figures: lines. We can see a set of points, which have the same one coordinate. We can see a set of points which have pairwise two distinct coordinates. These sets are lines in our geometry.

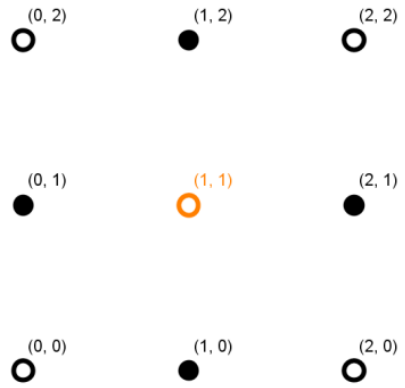


Line

The definition of the distance is more difficult than that of aline. Let P=(p_1, p_2) and Q=(q_1, q_2) be two points in the plane. Then we set dist(P, Q) := number of distinct coordinates of points P and Q. Thus this distance can be either 0, 1 or 2.



When we have the distance we can define more advanced figures. A circle with center O and radius r is the set of points in the plane whose distance from the center is r. We can see that every circle is built from only 4 points. We can see that radius of a circle can be only 0, 1 or 2.



Center: (1,1) Radius: 1
(Circle)

The next figure is a triangle. A triangle is formed by 3 points which are not collinear. A triangle forms a closed polygon. There are no other triangles than isosceles because the distance between 2 from 3 points will always be equal.

The next type of figures are quadrilaterals. The rhombus is a figure which has all sides of equal length, and the beginning of one side is the end of the next one. Something what is "prettier" than rhombus is a square.

A square is a quadrilateral that has all sides of equal length and it has all measures of angles the same, and the beginning of one side is the end of the next one. The definition is close to rhombus, but a square should have all angles the same. The square of the hypotenuse is equal to the sum of the squares of the other two sides. This property cannot be realized on our finite plane. Why? When we have a triangle with sides 1 and 1, the hypotenuse should be $\sqrt{2}$ ($1*1+1*1 = \sqrt{2}$) but it's 4… In our geometry there are only three numbers: 0, 1, 2. We have to 'convert' 4 to one of these numbers. So 4 mod (3) = 1. So one is not two. The reason why it can't be correct is quite obvious. The length of a segment in geometry modulo must be an integer.

This is just the beginning of the story. The geometry of the 9 points provides many opportunities to study well known objects from the Euclidean geometry in a new surroundings. And there are finite geometries which are richer. For example that of a plane over a field with 5 elements. But this is a different story…

THE PROBLEM OF DIVISION INTO "EXTREME AND MEAN RATIO"

Elina Andritsou and Ioannis Alonistiotis
Evangeliki Model School of Smyrna, Athens, Greece
Supervisor: Kalliopi Siopi

ABSTRACT

In our work we discuss the problem of *division into* "extreme and mean ratio" *in terms* of historical origin and evolution. Initially*, we refer to* the our study related proposals of Books II, IV, VI and XIII of the Euclid's Elements and present the evidence of two main proposals of the problem (II.11 and VI.30). Furthermore, we examine the relationship of the properties of this ratio according to the Golden Section problem. Finally*, we are looking for* when and where the Golden Section is used and under which prism.

INTRODUTION

"Geometry has two great treasures. One is the theorem of Pythagoras, the other is the division of a line into extreme and mean ratio (Golden section). The first we may compare to a measure of gold, the second we may name a precious jewel."
Johannes Kepler

THE PROBLEM

The origin is not historically verified. Some historians attribute it to the Pythagoreans and connect it to the study of the equation $x^2+ax= a^2$ as shown in geometric language in the Book II of Euclid's Elements , or the discovery of asymmetry in Ancient Greece, and others connect it with the construction of the pentagon from Theaetitos (415 B.C. – 369 B.C.).

*The division of a line AB into two segments using a point C so that, the large segment AC and the small segment CB with the whole of the line AB form equal ratios it is known as golden ratio, divine ratio, golden mean, golden section and Phi (**φ**).*

$$\frac{AB}{AC} = \frac{AC}{CB}$$

This terminology is similar to what the Euclid of Alexandria (≈300 BC) defines the division into "extreme and mean ratio" in Book VI of the Elements. The existence of the point that divides "a straight line" to "extreme and mean ratio" and the geometric division (construction), are described in the proposals II.11 and VI 30 of Books II and VI of the Elements of Euclid.

The problem of "division into middle and end ratio" that appears from Book II, IV, VI and XIII of Euclid's Elements is associated with 'parable passages', also appears on the construction of a pentagon, icosahedron and dodecahedron and is used in proportions theory. The concept of division in middle and end ratio did not end with the Euclid's Elements, but continued to play an important role in the development of mathematics and is directly related to the problem later in the 19th century, called the problem of the golden mean with which the equation is geometrically resolved, which positive root is related to the number $\phi = (\sqrt{5} + 1) /2$.

THE PROBLEM IN THE *ELEMENTS* OF EUCLID

- Several terms are referred to the problem we are discussing. Therefore we present some initial useful terms related to the problem of our study as:

  • A quantity x is called average ratio of two quantities a and b if the relationship $\frac{\alpha}{x}=\frac{x}{b}$ or equivalent $x^2=\alpha\cdot b$ applies.

  - The average proportional is constructed twice in the Elements of Euclid, in the proposal II.11 and the proposal VI.30, where it was first reported as mean and extreme ratio.

    • Two segments a and b with a> b are in 'extreme and mean ratio' if $\frac{\alpha}{b}=\frac{b}{\alpha-b}$ or equivalent $\alpha^2=\alpha\cdot b+b^2$.

In the second book of the Euclid's Elements is a set of proposals that are essentially geometric formalities of algebric formulas. The studied figures are always segments. Instead of the "product" we say "the rectangle contained by the a and b" and instead of α^2 we have "from the α square."

The proposals of the Elements associated with the problem of division into "extreme and mean ratio" are the II.11, VI.30, IV.10, IV.11, XIII. 16, XIII. 17 of Book II, IV, VI and XIII. From these we will present the evidence of proposals II.11 and VI.30 concerning the geometric construction of the subject of our study.

Proposition II.11 (1st construction): *To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.*

Definition VI.3: *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*

Proposition VI.30 (2st construction- golden section): *To cut a given finite straight line in extreme and mean ratio.*

Proposition IV.10: *To construct an isosceles triangle having each of the angles at the base double the remaining one. (golden triangle 72^0-72^0-36^0).*

Proposition IV.11: *To inscribe an equilateral and equiangular pentagon in a given circle.*

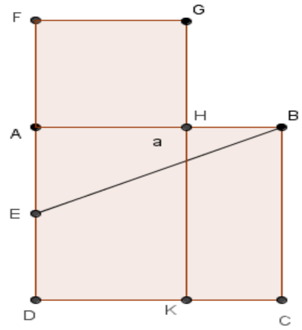
Proposition XIII. 16: *To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the icosahedron is the irrational straight line called minor.*

Proposition XIII. 17: *To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the dodecahedron is the irrational straight line called apotome.*

In the proposal II.11 is referred the expression "extreme and mean ratio" because it relates immediately with the construction of regular pentagon inscribed in a circle of radius R, namely the division of the circle into five equal arcs [Proposals IV.10 and IV.11]. There, it is proved that the side and diagonal of regular pentagon is also a product of self-similar golden mean. In the definition VI 3 is defined as the point that divides the straight line to end and average ratio. In proposal VI.30 now, again II.11 is proved in a construction way (as a special case of a parabola over hyperbola) and this proposal immediately in Book VII, is producing numbers. In Book XIII properties of the mean and the average ratio are proved [Proposals XIII.1 to XIII.6] because the icosahedron and a dodecahedron are related to regular pentagon.

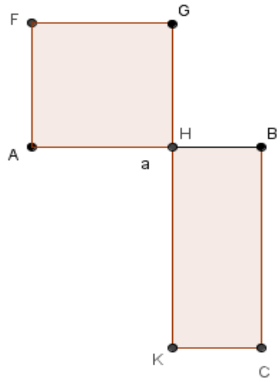
Proposition II. 11 (1st construction)
To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

Proof



We are asking to find point H of AB so that $\mathrm{AB} \cdot \mathrm{HB} = \mathrm{AH}^2$
We construct a rectangle ABCD and the middle E of AD. We construct BE and in the extension of DA we take EF=EB and construct the rectangle AFGH. We will prove that the point H is the asking one.
We extend GH, that cuts DC in K. According to the proposal II.6, the rectangle which is defined by DF and AF together with the square of side AE will be equivalent to the square of side EF, ie:

- $DF \cdot AF + EE^2 = EF^2 (1)$
- $DF \cdot AF + AE^2 = EB^2$ (2) as $EF = EB$
- $DF \cdot AF + AE^2 = AE^2 + AB^2$
(proposal I.47, Theorem of Pythagoras in the triangle EAB)
- $DF \cdot AF = AB^2$
- $(DFGK) = (ABCD)$ (3)



Abstracting from (3) the (AHKD), will have
$(AFGH) = (HBCK)$ or $AH^2 = BC \cdot HB$ or
$AH^2 = AB \cdot HB$ or $\frac{AH}{HB} = \frac{AB}{AH}$

Comments:
In the proposal II.11 the construction takes place in order to divide a given part of a line "α" in such a way that the rectangle with sides "a" and a part of "a", to be equal with the square which have its side the part of "a". This relationship is expressed by $\alpha \cdot (\alpha - b) = b^2$
The proposal II.11 is used in the proposal VI.10.

Proposals of the Elements that are used

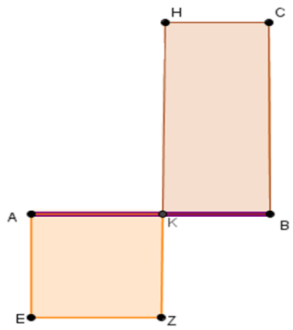
- Proposal I.47 (Pythagorean Theorem): In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
- Proposal II.6: If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.

Proposition VI.30

To cut a given finite straight line in extreme and mean ratio.

Proof
With side "a" we construct a square ABCD (I.46).
We extend DA and construct parallelogram DHZE which is equal to ABCD in a way that the per exaggeration parallelogram AKZE to be similar with the parallelogram ABCD (Proposal VI 29).

As (ABCD)=(DHZE) and if abstracting the parallelogram ADHK that is common to both, then (AKZE)=(BCHK).



But the parallelograms AKZE and BCHK have the same area and equal angles, so according to the proposal VI.14 the sides that contain the equal angles are reversal similar and $\frac{HK}{KZ}=\frac{AK}{KB}$

But $HK = AB$ and $KZ = AK$,

so $\frac{AB}{AK}=\frac{AK}{KB}$.

And as AB> AK it will be AK>KB.

Comments

According to Heath, the construction is an immediate application of the proposal VI.29 in the special occasion that the excess of the parallelogram is a square. This issue combining with the VI.30 is enough proof that this construction is by Euclid.

The proposal VI.30 is used in proposals 1,2,3,4,5,6,7,,8,9,10,11,16,17,18 of XIII Book of Euclid's Elements.

Proposals that are used:

- Proposal I. 46: To describe a square on a given straight line.
- Proposal VI. 14: In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.
- Proposal VI. 29: To apply a parallelogram equal to a given rectilinear figure to a given straight line but exceeding it by a parallelogram similar to a given one.

THE PROBLEM IN MORDEN TEXTBOOKS OF CEOMETRY (IN CREECE)

To divide a segment AB, into two unequal segments AC and CB in a way that the biggest of these to be in an extreme and mean ratio to the other and to the AB.

Proof:

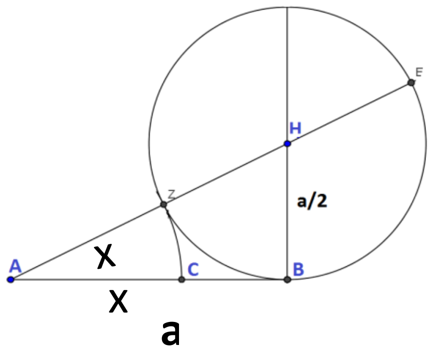
Lets have $AB = a$ and a point C with $AC = x$ the largest one. Then $CB = a - x$ and it will be $\frac{AC}{AB}=\frac{CB}{AC}$ (1)

or $AC^2 = AB \cdot CB$

or $x^2 = a \cdot (a-x)$ (2).

The relation (2) can be written $x^2 + ax - a^2 = 0$ or $x(x+a) = a^2$ (3)

In order to find a point C in the segment AB so that $AC = x$ we draw a circle $\left(H, \frac{\alpha}{2}\right)$ that osculates to the segment AB in the point B and construct AH that cuts the circle to Z and E. Then



$$AB^2 = AZ \cdot AE = AZ \cdot (AZ + ZE) = AZ \cdot (AZ + AB)$$

$$or\ \ a^2 = AZ \cdot (AZ + a)$$

So, the segment AZ has the right length and the point C will be the sectional of the circle $\left(A, \frac{\alpha}{2}\right)$ and the segment AB.

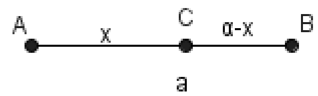
Comment

In these constructions the proposals III.36 και III.37 του III of the Books of Euclid are used.

- Proposition III.36: If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.
- Proposition III.37: If a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.

THE PROBLEM AND GEOMETRICALLY SOLVING THE 2nd DEGREE EQUATION –THE NUMDER *Phi (φ).*

The problem of division of a segment AB from a point C into extreme and mean ratio today is known as Golden Section and the ratio $\frac{AB}{AC} = \frac{AC}{CB}$



or equivalent $\frac{\alpha}{x} = \frac{x}{\alpha - x}$ is called Golden Ratio or Divine Ratio and is symbolized by the greek letter tau (τ). The symbol (τ) means "the cut" or "the section" in Greek. With the problem of Golden Section we can geometrically solve the equation $\alpha(\alpha - x) = x^2$ or $x^2 + \alpha x - \alpha^2 = 0$ which the root is the number $x = \frac{a(\sqrt{5}-1)}{2}$.

Then the golden ratio becomes $\frac{\alpha}{x} = \frac{x}{\alpha - x} = \frac{\sqrt{5}+1}{2}$.

During 19th century, American Mathematician Mark Barr gave to golden ratio a new name *Phi (φ).*

This symbol φ becomes from the $\phi = \frac{\sqrt{5}+1}{2}$ Greek sculpture of Classical Ancient Greece Phidias, because we can meet the Golden Ratio to many of its works like Parthenon.

Golden Ratio is an irrational number. Many historical believe that this irrational has been found by the Pythagoreans' about 5th b.C. who believed that irrationals was something like a cosmic error.

Comment

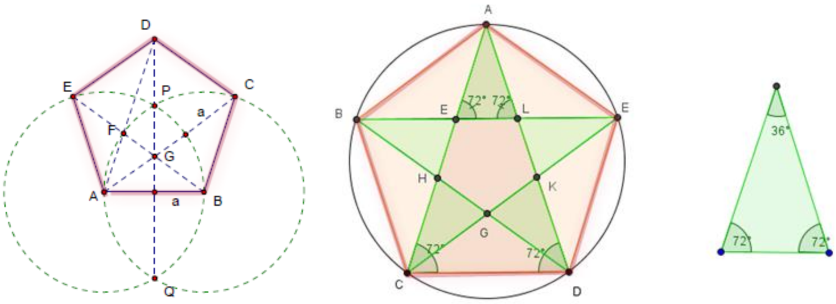
This problem or otherwise the geometrical solution of the equation $x^2 + \alpha x - \alpha^2 = 0$ is described using the proposals III.36 and III.37 of the book III of Euclid's.

THE PROBLEM IN ANCIENT GREECE

Plato prophesied the importance of the golden ratio long before Euclid describe the details and saw the world in terms of perfect geometric proportions and symmetries, in the forms of the five Platonic solids, the tetrahedron, the cube, the octahedron and the icosahedron. The platonic solids are the only solid that all the seats are regular polygons and each vertex is convex. Each of these solid entered a sphere with all their vertices on it. Every seat in the Platonic solids are regular polygon: equilateral triangle, square, and regular pentagon. The tetrahedron consists of four equilateral triangles, the cube of six squares, the octahedron has eight seats in the form of equilateral triangles, the dodecahedron has twelve regular pentagons as seats and icosahedron twenty equilateral triangles. The dodecahedron and icosahedron are very interest. If any of them constructed with edge length of a unit, it is very easy for someone to recognize the important role played by Colden Ratio to their dimensions.

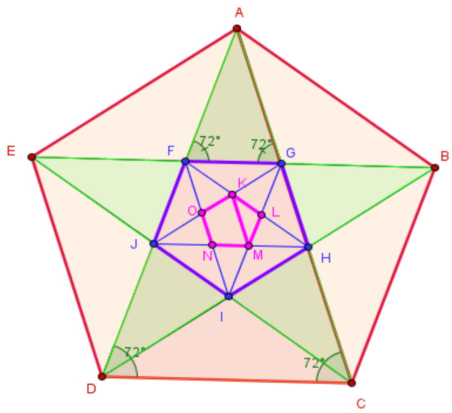
	Surface area	volume
dodekahedro	$15\varphi/(3\text{-}\varphi)$	$5\varphi^3/(6\text{-}2\varphi)$
icosahedro	$5\sqrt{3}$	$5\varphi^5/6$

The construction of the pentagon with ruler and compass is very interesting, as shown properties of extreme and mean ratio between the sides and diagonals.
The side and the diagonal of the regular pentagon is also a product of self-similar of the Golden Section.



The diagonals of a regular pentagon intersecting around the center of the pentagon, form another smaller regular pentagon, etc. in an infinite sequence of regular pentagons that everyone is into another. The sides of all these pentagons and diagonals of the pentagons are permanently in a golden ratio sequence-based on φ. Each diagonal of the regular pentagon is intersected by another at the site of the golden ratio, which means that each diagonal divides the one that intersects and is divided by it, in extreme and mean ratio. The ratio of any diagonal of a regular pentagon to the side of this pentagon is the golden ratio φ.

$$\frac{FG}{KM} \approx \varphi$$
$$\frac{AB}{FH} \approx \varphi$$
$$\frac{DB}{BC} \approx \varphi$$
$$\frac{BE}{BF} = \frac{AD}{DF} \approx \varphi$$



THE PROBLEM AFTER *EUCLID*

After Euclid problem of division in extreme and mean ratio appears the so-called "Supplement" or Book XIV of the Elements attributed to Hypsicles of Alexandria (2nd BC). Also, it appears in the work of Hero of Alexandria and is related to the determination of the surface of the pentagon and the dodecagon, and the "synagogue" of Pappus of Alexandria in the construction of the icosahedron and dodecahedron and the comparison theorems of their volumes.

In Arab tradition there are indications introduction of the concept of division of a segment in medium and extreme ratio, although the works of some Arabs considered related problems, such as Al-Chouarizmi (about 780-850 AD) and Abul -Ouafa (ca. 940-997 AD).

In the European tradition, the origins of the study of the properties of division in medium and extreme ratio are stemming to Leonardo of Pisa or Fibonacci (ca. 1180- 1250 AD), who examines metric problems of the pentagon and the dodecagon, and the identification problems volume icosahedron and dodecahedron. Fibonacci with the "problem of the rabbits" is creating the following sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... that is called sequence of Fibonacci and is formed according to the rule:

$$u_0 = 1,\ u_1 = 1,\ u_n = u_{n-1} + u_{n-2}$$

The relationship of this sequence with the problem of division in medium and extreme ratio is that the limit of the ratio of the next last term of the sequence is the value of the medium and extreme ratio, ie the root of the equation, $x^2 + \alpha x - \alpha^2 = 0$ ie th $\phi = \frac{\sqrt{5}+1}{2}$ er There are indications that the Fibonacci knew this relationship, which oc er mathematicians of 16th century, Kepler, Zizar, Simpson.

In 15th-16th century interest in the division in middle and extreme ratio revitalized compared with its applications in geometry and architecture. In this context introduced the term 'golden ratio' by Leonardo da Vinci. In 1509 the project issued "The divine proportion", which although is specially dedicated to the problem of division in "extreme and mean ratio".

CONCLUSIONS

The way that the structure of the division is done, varies between that shown in Figures Euclid and subsequent mathematical as Descartes. For the ancient Greeks the construction of the golden section had not the usefulness of geometric equation solving, but looked more like a constant ratio with application to construct regular polygons in books VI and XIII. The constructions are based on equations and similar areas of rectangles and squares as the product of two line segments is surface, and not straight section as in Descartes. Considering some basic elements of the presence of the 'golden mean', then perhaps understand its greatness. Initially, everywhere eg. in nature, the universe, the human body, in order to impart harmony. Additionally, it is a great discovery for both the science of mathematics and to the development of human history, as it was triggered, so that people begin to perceive the world and life with another eye. That is to feel that they can explain this inexplicable perfection of life and nature through a "divine" number.

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The future of mathematical communication development in the communication of mathematics and other sciences

Kosta Balać
Elementary school ''Ivo Lola Ribar'', Sombor, Serbia

Aleksandar Brzaković
Elementary school ''Ivo Lola Ribar'', Sombor, Serbia

ABSTRACT

This paper work contains the method and the development of mathematics and science through the ages. This work will be presented examples and methods of communication as a mathematician and communication between mathematicians and physicists, mathematical communication between peoples, development and transfer of bookmarks in the SI system, the development of mathematics throughout history, the connection between mathematics and physics, communication with aliens, transfer the mathematical content of the next generation, the way cooperation with other math related sciences. There will be a couple of funny stories in the field of mathematics.

INTRODUCTION

CHANGES ARE SOMETHING INEVITABLE IN LIFE. THEY HAPPEN EVER DAY, EVERY HOUR, EVERY MINUTE, EVERY SECOND. WITHOUT CHANGES, LIFE WOULD NOT BE LIFE.

Every change in communication has been the cause of lifestyle change. Besides the verbal communication, there is also the nonverbal one, as well as the communication through mechanical and electromagnetic waves. For example, it is well-known that animals communicate with each other by ultrasound and infrasound, and the application of radio waves has enabled people to use mobile telephones, personal computers and other appliances without which the modern life would be unimaginable.

Throughout time, people have been improving not only the spoken language, but the scientific language as well, i.e. the language of mathematics. Why is mathematics so special within this? It is because every scientific discovery has appeared in cooperation with mathematics.

The beginnings of mathematics are found in the distant past, even down to 35000 BC. The discovery of the baboon bone with 29 cuts is something well-known. It is believed that this is a real proof of the beginnings of calculating left by the early humans.

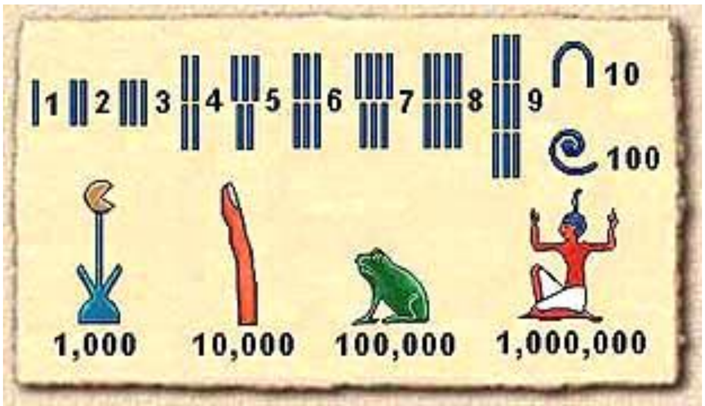
Through this work I will show you in what way mathematics has influenced the change of communication, and by this the change of people's lives. I will also remind you of the mathematical messages which we send beyond our planet in order to contact extraterrestrials intelligences.

This work also displays the connection between mathematics and physics because these two sciences are close and they use the same language.

HISTORY

Mathematics was not always used by all nationalities. Before the modern times and the spreading of knowledge only some nationalities knew and applied mathematics. Some very interesting facts are known about Greek, Babylonian and Egyptian mathematics. The oldest examples of mathematical development are Plimpton 322(1990BC), Rhind Mathematical Papyrus (2000-1800BC) and the Moscow mathematical Papyrus (1890BC). All these writings deal in one part with the Pythagorean Theorem, and it is no wonder that teachers at school insist on the knowledge of this famous theorem. There is almost no task in geometry where we do not apply the Pythagorean Theorem. And it is not only the mathematical task, it is the physics tasks as well. For example, when we study optics or put forces into concordance.

The Chinese had a very interesting way of writing numbers. For example, the number 123 used to be written by writing number 1 first and then the mark for number 100, after that number 2 and the mark for number 10 and finally number 3.

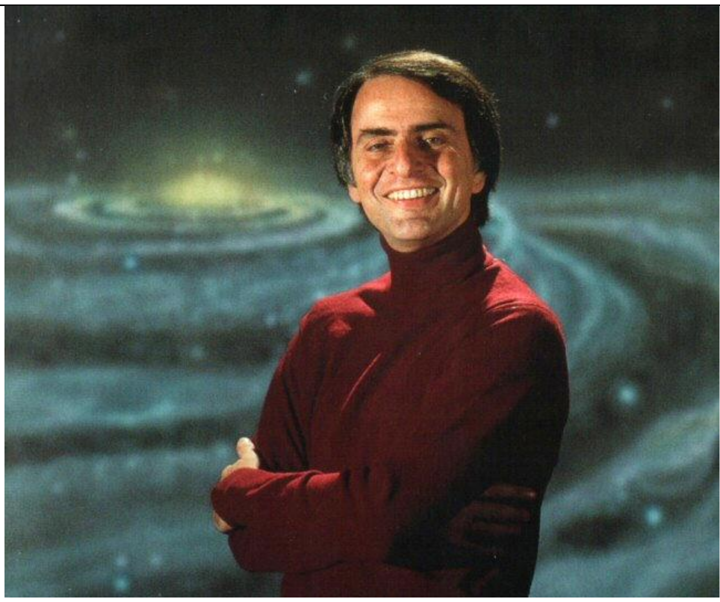


Picture 1

Communication with aliens

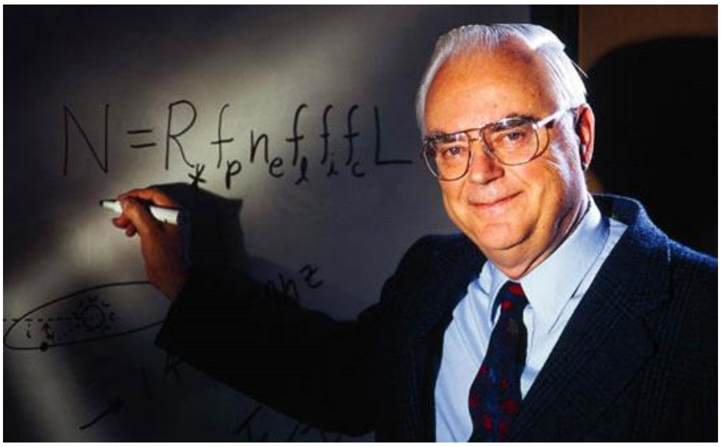
If we ever get into contact with some extraterrestrial intelligence, we will most probably use the language of mathematics. COMMUNICATION WITH EXTRATERRESTRIAL INTELLIGENCE is an area which deals with writing and sending messages, with the help of which we could establish contact with beings from other planets. Sending these messages has had the aim to determine where we are in space. Besides this, our wish is to get into contact with extraterrestrial beings in order to improve our life. The messages contain information about what we look like and what technology we have. Apart from this, the question that remains is whether the messages will ever reach some new civilization that will be able to understand them.

There is a famous message, written by Frank Drake and Carl Sagan. It is called the Acreibo message. That message is made of 1679 pixels, 73 rows and 2 columns. It contains numbers from 1 to 10, atomic numbers of hydrogen, carbon, nitrogen, oxygen and phosphorus, formulas of sugar and DNA, a picture of a human being, his height, even the number of people that lived on Earth at the time when the message was sent. In the 19^{th} century there were books and articles about possible life forms on other planets. People have always believed that “conscious” beings live in other places of the Universe as well. At that time, traveling into space was not possible, but people used to send messages into space even before the invention of the radio. Carl Friedrich Gause suggested the creation of a giant drawing the Siberian tundra in the shape of a triangle and 3 squares which represented the Pythagorean theorem. He hoped to establish contact with aliens in that way. (IMGUR, ACREIBO, PICTURE FROM TWEETER)



Picture 2

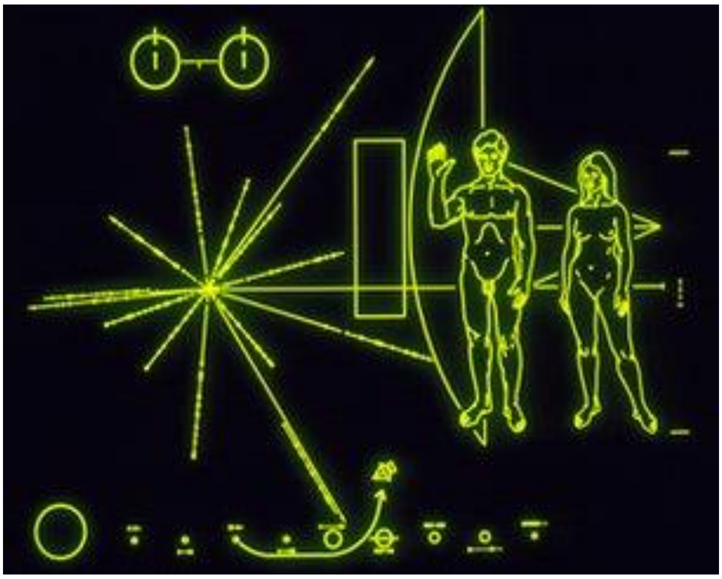
CARL SAGAN : He was an American astronomer, astrobiologyst and a populisator of science, writer if popular science books. He was born in Brooklyn in New York.



Picture 3

FRANK DRAKE He was a an American astronomer and astrophysist. He is most famous for participating in the SEAT project in 1960. as well as for the Drake equation.

The two of them worked together on the ACREIBO message in 1974.



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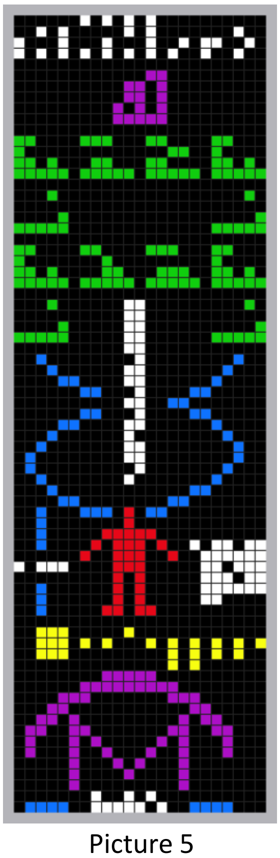


Picture 5

Mathematics and physics

There is an obvious connection between mathematics and physics. It would be impossible to solve a problem from physics without mathematics. We use formulas, i.e. the language of mathematics in order to describe natural phenomena that we study within physics classes.

Most of the famous physicists have been mathematicians, professors of mathematics or scientist. For example, Isaac Newton taught mathematics, and was a famous physicist. And, in order to avoid misunderstandings, a unique system of VALUES and their measuring units was introduced.
I will show one example where there is a unity in marking SIZES AND MEASURING UNITS which are used by both mathematicians and physicists.

TASK: Calculate the DENSITY of the material of which a cube is made whose sides are 5cm if its MASS is 20kg

a=5cm=0,05m
m=20g=0,02kg
ρ-?

V=a·a·a
V=(0,05cm)3
V=0,000125m3
ρ=m/V
ρ=0,02kg/0,000125m3
ρ=160kg/m3 (a good density chart)

From a large number of examples I have taken the SI system as an example of the communication between mathematics and physics.

SI SYSTEM (THE INTERNATIONAL SYSTEM OF UNITS) is the most widely used system of units. It is the most common system for everyday trade in the world, and it is almost universally used in science. Before the SI system the MKS system (meter-kilogram-second) and the CGS system (centimeter-gram-second) were used.

TABEL

Physical quantity measured	Base unit	SI abbreviation
Amount of substance	mole	mol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Thermodynamic temperature	kelvin	K
Electric current	ampere	A
Luminous intensity	candela	cd

This system which contains 7 physical quantities and their measuring units is one of the best scientific medicines that we use to cure misunderstandings.

Imagine some Europeans, adventurists, who meet a monk somewhere in Tibet. Exhausted and tired, they ask themselves how much time they need until they arrive at the closest populated place. The answer they got was: 12 CUPS OF TEA! The time the monk needs for drinking one cup of tea is his measuring unit of time. From the monk`s point of view this is the only appropriate way of measuring time, but the Europeans do not understand it.
The greatest confusing on the planet has been created by the measuring units for length. I will present just some of those that I have found: meter, thumb, foot, inch…

There was really a great need to introduce order into this as soon as possible. I also believe that the use of the Celsius, Kelvin and Fahrenheit scales for temperature measurement is also a well-known example. What is important is that not only scientist but also all the other people

know the way of converting one unit into another. In Serbia we use the Celsius scale, but at school we do exercises where we have to stick to the units of the SI system.
Because of this I know that 20˚C=293˚K. The procedure is simple. The Celsius 0 is at 273 Kelvin`s degrees. And because of this we just add the number of the Celsius degrees to this number.

Where does mathematics originate

A lot of people is asking themselves the question whether it is still necessary to have mathematicians for mathematics. Herbert Simon, the winner of the "Nobel Prize" for economics in 1978, and Allen Newel were working on the creation of a programme called the General Problem Solver. The authors announced it as a "programme that simulates human thoughts". Joseph Bajcenbaun started to doubt the great confidence people had in computers. Giancarlo Riotta wrote: The computer is just a means that helps us to finish something faster which we could do without it anyway. But where does mathematics start its life? (The example from the book CAN I COUNT).
There is not any precise answer to this question. In most of the cases, mathematics is not the result of any planned process of thinking. It happens very often that something comes to our mind and this something could be the solution or a new beginning that we have been looking for. Mathematics does not originate only in laboratories. It also appears at the café, in bed and even at prison.
At the café: Sitting for hours with some people does not sound like a productive piece of work at all. However, some cafes and its regular guests mathematicians have become famous and significant. Some good ideas can come up at cafes, but to develop them we need some piece and quietness, and there comes the prison. Just think about it, there is silence, nobody is disturbing you and you can focus easily on something and get into the process of thinking. A good example for this is the Frenchman Jean Victor Poncelet who created projective geometry while he was imprisoned in Russia. This proves the fact that mathematics is all around us, and at places where it does not exist we have the power to create it.
Mathematics will always move on and computers will be more and more in use, but only as a tool in the hands of mathematicians from who everything begins.

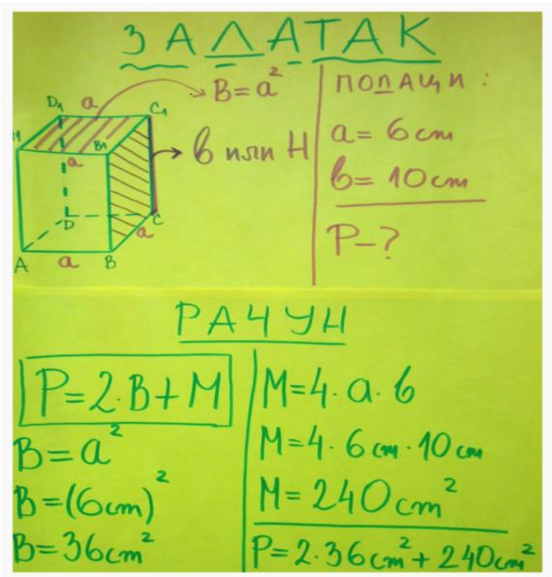
MATHEMATICS AND SOCIAL NETWORKS

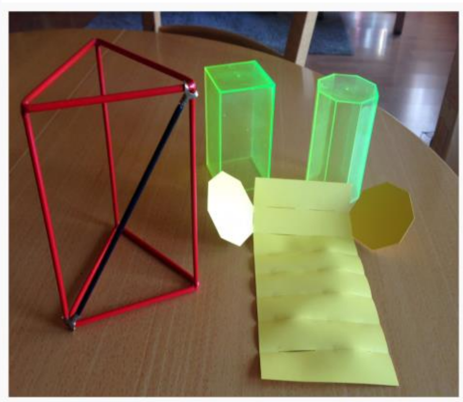
Learning of all subjects at school, not only mathematics, is moving towards the Internet, especially towards social networks. My school friends and I use our teachers` blogs, the school web page and educational pages in order to study faster and easier and to be informed. In Serbia we have a really difficult final exam at the end of the 8^{th} grade, before entering secondary school. In order to prepare this exam successfully we are constantly in contact with each others and we use web pages for the exam preparations.

As an example of peer learning we will introduce the mail experiment that we have realized with a student from Slovenia. The aim of the experiment was learning and discovering of new mathematical topics by using Skype.

Picture 6

28.2. 2015. I realized an experiment with a friend from Slovenia. Via Skype I tried to tell her mathematical content in the area of the prism. The aim of this study is the ability to transfer knowledge among peers, thanks to social networks. For the experiment I was first prepared. I got models and I made posters. During the experiment, my friend and I we encountered some problems. For a short time it took to adopt and understand a lot of formulas, information, concepts. In addition there is a difference in the programs so sorry about that communication was difficult.





Pictures 7 and 8

This is a simulation of online learning, and resembles what scientists are doing around the world. I am satisfied with what has been learned Lea. when we encountered some difficulties, I am helping her and directed. Finally, we solve all the tasks that I had planned. Also, the time it took to create these tasks is less than school class.

In this way we can be on different sides of the world and to learn together. Also learn fast because as peers better understood.

Mathematics in the future

Tom Gauer started a discussion about "massive cooperation" on his blog on 27. January 2009. He wanted one group of mathematicians to work together on finding the solution of some particular problem. His aim was to use this big number of mathematicians in order to create one virtual "supermathematician" whose brain is the sum of all the mathematicians` brains who are at work together. He needed 6 weeks and over 1000 replies until success was reached. After his success a question arises: "what after that"?

That "supermathematician" can do something that we have seen only in science fiction movies. He solves all the tasks that teachers give us, he creates problematic situations and teaches us how to solve them. He uses mathematics for creating software, crates various applications and video games.

He could become our virtual assistant with the help of who we could solve most of the problems, if it is possible. Will we need special servers or could we use them through the Internet?

Apart from this, his task would be to form tests which would enable him to have an insight of my mathematical weaknesses. His task would be to solve problems and to enhance my skills. In that way I would become a supermathematician as well.

Different supermathematicians of my peers would be able to communicate with each other. That would be some sort of their virtual world where they would exchange data and help us, the kids.

Teachers do something similar, but they are not constantly available. Sometimes it happens that we are absent from school because of illnesses and because of that we miss important lessons. With the help of our supermatematician that problem would be solved.

A really important role of my supermathematician would be to, according to my demands, create video games and simulations of some real life situations that I cannot experience every day, for example diving in extreme depths, staying in space stations, paragliding and many others.

Reference material:
May I Count? – Gunther Ziegler
A Passion for Mathematics – Clifford Pickover
The Kingfisher Science Encyclopedia
and many internet sites

Translate: Violeta Iđuški , English teacher in Elementary school ˝Ivo Lola Ribar˝, Sombor, Monoštorska 8, Serbia