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Gregory Makrides, Yiannis Kalaitzis

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CONTACT DETAILS

Contact Email
info@euromath.org

Contact Phone
_00 357 22 283 600_

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MP26. GEOMETRICAL STRUCTUR OF SANTUR AND BAGPIPE FROM VIEWPOINT OF MATHEMATICS

Parnian Shojaei, Parmida Azizi, Ava Ilami
Farzanegan Highschool, Bushehr, Iran

ABSTRACT

This project is devoted into the two most important sections. Firstly, plenary description of Bagpipe, such as: History, the components of constituent its modulation involves voice leading, specifically discusses on some kinds of topological metrics of musical data related to bagpipe, an abridged history and introduction of the Iranian version of bagpipe called Ney-anban. Secondly one of the eldest classical instruments called Santour which is of the main instruments of Iranian classical music is introduced and some of the mathematical points of its structure are showed.

BAGPIPE

The Highland bagpipe, considered ‘the national instrument of Scotland, is one of the most recognized and celebrated icons of traditional music in the world. It is also among the least understood by Western musicians and ethnomusicologists alike. But this is beginning to change: Scottish bagpipe music and tradition – particularly, but not exclusively, the Highland bagpipe – has enjoyed an unprecedented surge in public visibility and scholarly attention over the past two decades. Founded mainly on the work of Robin Lorimer, Peter Cooke and Francis Collinson of the Edinburgh school in the 1960s and 1970s, an increasingly eclectic and practice-based corps of scholarship has emerged, bringing to light, as never before, treatment of evidence subject to the performer’s own grasp of his craft. Scholarship has also become increasingly international and reflexive in scope. Bagpipes have been promoted as ‘national’, ‘cultural’ or ‘regional’ instruments – in Scotland, Galician Spain, Brittany, Bulgaria and Ruthenia – and the meanings and political overtones of this need to be more deeply explored. Bagpipe revivals, where an instrument has become extinct.

_NEY_ANBAN_

The Iranian version of bagpipe which names Ney_anban, is played in southern Iran. This version of bagpipe even does not have a complete octave (!) but this instrument has a warm sound and most of people enjoy hearing its sound. When old sailors were in a far distance from their home and family, ney_anban and traditional songs and dances helped them to be strong. Obviously this instrument is not so famous and there is not a regular method for it but musicians are attracting to this instrument more than before. There is a picture of ney_anban below.

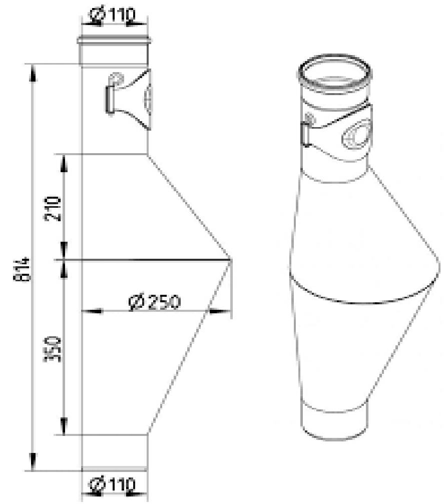
SANTOUR

Santour is an originally Iranian instrument and it is one of the eldest instruments of Asia. This instrument was traded and travelled to different parts of the world and each country customized and designed their own versions to adapt to their musical scales and tunings. The original Santour was made with tree bark, stones and stringed with goat intestines. Santour is also the father of the harp, the Chinese Yangqin, the harpsichord, the Qanun, the cimbalom and the American and European hammered dulcimers. Santour has accepted many changes in different periods of time. And the instrument we call Santour today is the instrument which has been used since about a century ago.
The oval-shaped Mezrabs (mallets) are feather-weight and are held between the thumb, index and middle fingers. A typical Persian Santour has two sets of bridges, providing a range of approximately three octaves. The right-hand strings are made of brass or copper, while the left-hand strings are made of steel. Two rows of 9 bridges a total of 18 vaults divide the Santour into three positions. Over each vault cross four strings tuned in unison, spanning horizontally across the right and left side of the instrument. There are three sections of nine pitches: each for the bass, middle and higher octave called behind the left vault comprising 27 notes altogether. The top "F, M" notes are repeated twice, creating a total of 21 separated notes in the Santour. The Persian Santour is primarily tuned to a variety of different diatonic scales utilizing 1/4 tones which are designated into 12 organization (dastgahs) of Persian classical music. These 12 organizations are the repertoire of Persian classical music known as Radif.

THE RELATION BETWEEN BAGPIPES AND MATHEMATICS

One of the theorems which describe the Geometrical structure of Highland Bagpipe is called bagpipe theorem of Nyikos (1984). In mathematics the bagpipe theorem describes the structural the connected (but possibly non-paracompact) bounded surfaces by showing that they are “bagpipes” the connected some of compact ‘‘bag’’ with several long pipes.
We note that a space is called ω_bounded if the closure of every countable set is compact. For example the half-open long line is ω_boundedness is equivalent to compactness.

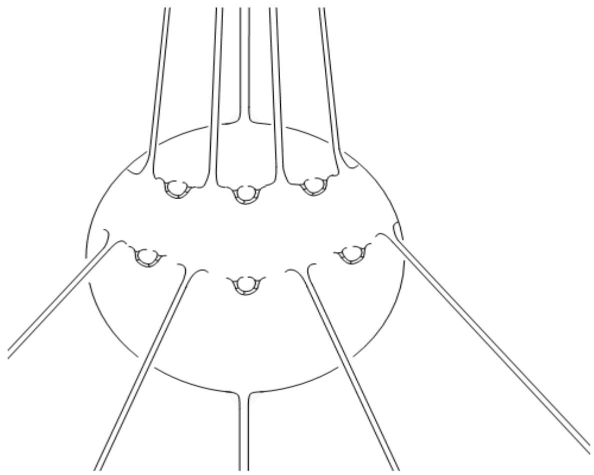
The bagpipe theorem states that every ω_bounded connected surface is the connected sum of a compact connected surface and a finite number of long pipes. A long pipe is roughly an increasing union of copies of the half-open cylinder. There are different isomorphism classes of long pipes. Two example of long pipes are the product of a circle with a long line (long at one end) and the long plane, a product of two long line which are long at both ends. With a disk removed.



HOMEOMORPHISM OF BAGPIPES

In this section we investigate the mapping classes group of an orientable bounded surface. Such a surface of splits, by the bagpipe theorem of Nyikos, into a union of bag (a compact surface with boundary) and finitely many long pipes. The subgroup consisting of classes of homomorphism fixing the boundary of the bag is a normal subgroup and is a homomorphism image of the product of mapping classes groups of the bag and the pipes.

In fact we may obtain a surface of higher genus and having more long pipes by spreading handles and mutually homomorphism long pipe arounds the equator, arranged in such a way that a rotation of the sphere through takes each handle and large pipe to the adjacent handle respectively long pipe. More bound of handles or both may be distributed along other lines of latitude. Of course the long pipe within a particular band must be mutually homomorphic.



THE CHROMATIC SANTOUR

The chromatic Santour is set up according to international music standards and includes all chromatic intervals. This Santour is similar to the original version and its difference is in the size and the number of vaults and strings. The purpose of creating this version of Santour is to perform world music with a completely oriental sound. However there is the ability to adjust the sound in a desired way and play original Iranian music melodies on the instrument.

SANTOURS WITH MORE VAULTS

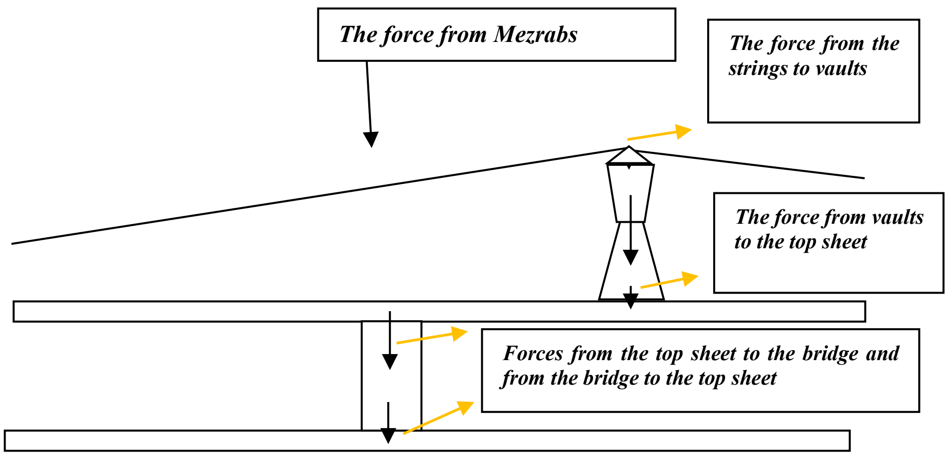
Due to the limited ability of the instrument for tracks of various organizations to be played on without tuning it again, Santours with more vaults has been invented. These instrument at most has fifteen vaults in each position and that gives the player a freedom to play more tracks of various organizations on the Santour.

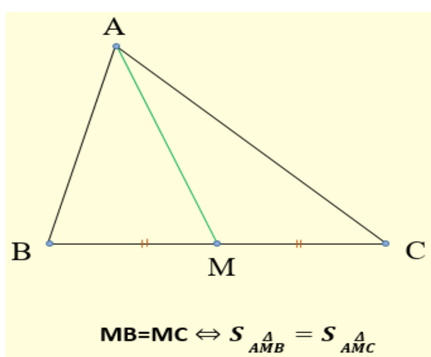
THE IMPORTANCE OF THE THIRD VAULT

In the planning of the dimension of Santour, strings have a suitable traction and the desired sound comes out. Because of the manner of bass and high strings' tremblement, the sound of strings which are closer to the bigger side of the trapeze are more bass than those which are further. Therefore, the determinate length of an (A-tuned) string is different from a (C-tuned) one. In a Santour, when the (G) note is embedded in the third vault, that Santour calls ((G-tuned) Santour). When the player seats in front of a Santour, in a normal situation, the mezrabs of the player stay on the third vault so the third vault is easier for the player to use it. And that is the reason of the third vaults importance.

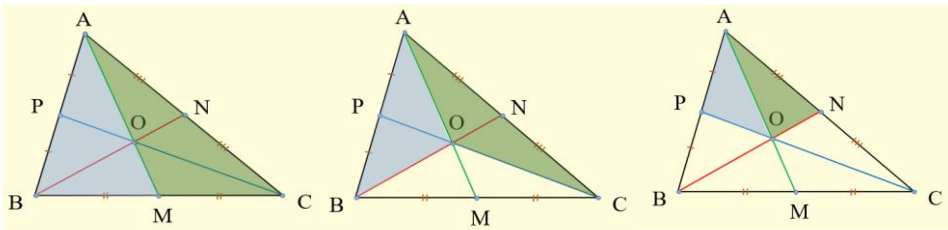
The geometrical structure of Santour

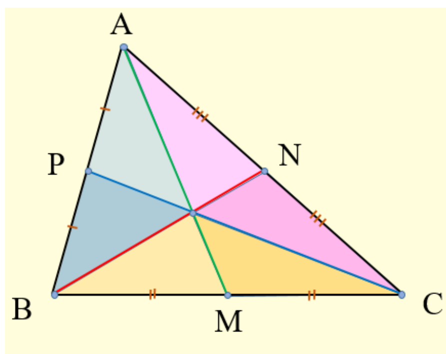
One of the most important parts of Santour are its bridges. An insignificant displacement in bridges, changes the whole sound of the instrument.
At first, bridges had placed inside the box of Santour in one piece. But today makers situate bridges with calculating the center of gravity of six triangles which cover the surface of the trapeze of Santour.
Depending on how sounds are played on the instrument, bridges must situate in a place with the same distance from vaults. And this is not possible except with situating a bridge for each two vaults, that according to the difference of presses of notes on sheets, the need for player to move the vaults and the influence of other vaults on strings that they are not relevant to, it does not give us the wanted quality.

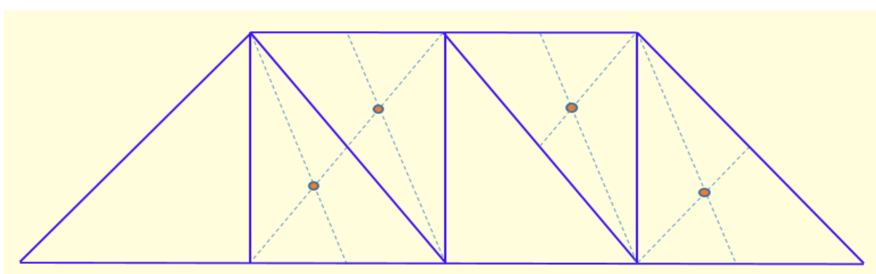




The centers of gravity of a triangle is the intersection point of its medians. The three medians of a triangle divide the surface of the triangle into six equal parts.







Therefore, if the centre of gravity is chosen as the location of bridges, the came-out sound would be closer to the desired one.
Two of triangles are ignored because there is not any vault on them.

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MP39. THE AREA OF SURFACES WHICH ARE BOUNDED BY THE "BLACK HOLE" FUNCTIONS

Bogdan Branisteanu, Darius Feher
"Mihai Eminescu" National College Satu Mare, Romania

ABSTRACT

We will define a class of functions which, because of their properties and their graphs, we named "Black Hole" functions.
In this paper we will find the area of surfaces bordered by these functions ("Black Hole" functions) and we will establish a correlation between these areas and Riemann Zeta function (ζ).

1. A Class of „Black Hole" Functions

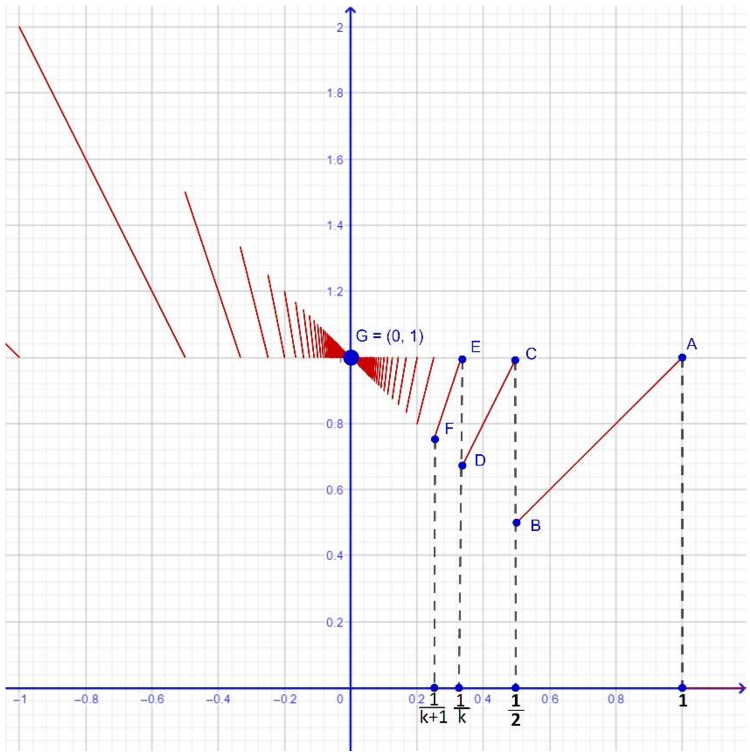
We define the „black hole" functions as functions whose plots are made of an infinity of arcs of the curves (or segments) which are „lost" (or are „absorbed") in a point which we call „black hole".

For example,

$$f:\mathbb{R}\to\mathbb{R}, f(x)=x\cdot\left[\frac{1}{x}\right], \qquad x\neq 0, \qquad f(0)=1.$$

where [a] is the integer part of a. We made some further investigations into the properties of this function and we saw that it can be redefined piecewise as:

$$f(x)=\begin{cases}-x & , if \;\; x\le -1\\ -(k+1)\cdot x & , if \;\; -\frac{1}{k}<x\le -\frac{1}{k+1},\; k\in\mathbb{N}^*\\ 1 & , if \;\; x=0\\ k\cdot x & , if \;\; \frac{1}{k+1}<x\le\frac{1}{k},\; k\in\mathbb{N}^*\\ 0 & , if \;\; x>1\end{cases}$$



We notice that the plot of the function constituted of the half-line which superposes with O_x for $x > 1$ and of infinite number of segments [AB) where $A(\frac{1}{k}, 1)$ and $B(\frac{1}{k+1}, \frac{k}{k+1})$ which are „absorbed" in the point G(0,1). For $x \in (-\infty, 0)$ the plot of function is number of segments for $x \in (-1,0)$ which are „absorbed" in the same point.

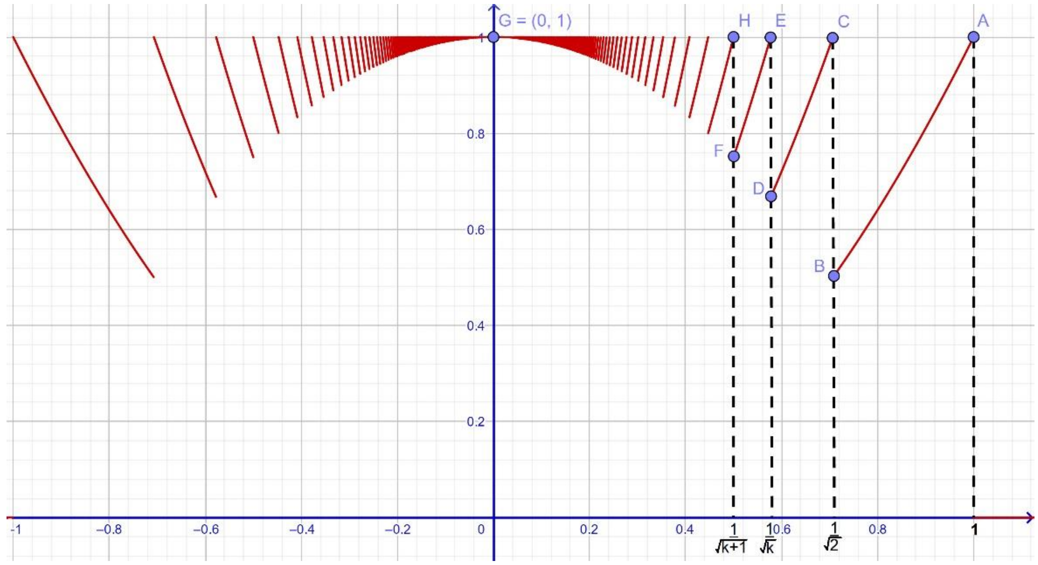
Remark 1. The set of the gradients of all segments is the set of integer number $\mathbb{Z}$.

Now, let us take another example. For instance:

$$g: \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2 \cdot \left[\frac{1}{x^2}\right], \ x \neq 0, \ g(0) = 1.$$

where [a] is the integer part of a. The function can be rewritten piecewise as:

$$g(x) = \begin{cases} 0 & , if \ \ x \leq -1 \\ -k \cdot x^2 & , if \ \ -\frac{1}{\sqrt{k}} < x \leq -\frac{1}{\sqrt{(k+1)}}, \ k \in \mathbb{N}^* \\ 1 & , if \ \ x = 0 \\ k \cdot x^2 & , if \ \ \frac{1}{\sqrt{k+1}} < x \leq \frac{1}{\sqrt{k}}, \ k \in \mathbb{N}^* \\ 0 & , if \ \ x > 1 \end{cases}$$



We can observe that the plot of the function is constituted of an infinite number of arcs AB where where $A(\frac{1}{\sqrt{k}},1)$ and $B(\frac{1}{\sqrt{k+1}},\frac{k}{k+1})$ which are also „absorbed" in the point G(0,1). For $x\in(-\infty,0)$ the plot of function is number of segments for $x\in(-1{,}0)$ which are „absorbed" in the same point. Both functions that we presented to you had a finite „black hole".

2. Calculating the Area of Surfaces Which Are Bounded by the "Black Hole" Functions Using Mathematical Series

In 1859 Bernhard Riemann published the paper "On the Number of Primes less than a Given Magnitude" in which he defined the zeta function, $\zeta(s)$.
The Riemann zeta function, $\zeta(s)$, is a function of a complex variable $s=\sigma+it$, σ,t ∈ $\mathbb{R}$. The following infinite series converges for any complex number s with the real part greater than 1, and defines:

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots,\sigma>1$$

The Riemann hypothesis is still considered to be the greatest unsolved problem in the history of mathematics.
Let us take another look at the function that we previously defined:

$$f(x)=2x\cdot\left[\frac{1}{x}\right],\ \ x\neq 0\quad and\quad f(0)=1$$

Now, we aim to calculate the surface of the area bounded by the function f between 0 and 1.
To do that we consider:

$$\frac{1}{k+1} < x \leq \frac{1}{k} \quad resulting \;\; that \;\; [\frac{1}{x}] = k$$

$$\int_0^1 2x \cdot \left[\frac{1}{x}\right] dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k+1}}^{\frac{1}{k}} 2k \cdot x \quad dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(k \cdot x^2 |_{\frac{1}{k+1}}^{\frac{1}{k}} \right) == \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{k}{(k+1)^2}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} + \frac{1}{(k+1)^2}\right) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} + + \sum_{k=1}^{n} \frac{1}{(k+1)^2}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{(k+1)^2}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(n+1)^2}\right) = \zeta(2) = \frac{\pi^2}{6}$$

This property helps us define a new subclass of „black hole" functions, which is:

$$g_s(x) = s \cdot x^{s-1} \cdot \left[\frac{1}{x}\right], \qquad x > 0, \qquad s \geq 1$$

We can observe that: $f(x) = g_2(x), \ \forall x > 0$

These functions have similar proprieties, the „black hole" is in G(0,0), for s>2 and it is ∞ *for* $s \in (1,2)$. *If s=2, the „black hole" will be in G(0,2).*

Theorem: $\int_0^1 g_s(x)dx = \zeta(s), \ s > 1.$

3. Particular cases of „black hole" functions with geometric interpretation

3.1 Functions with black holes at ∞

We will consider two new functions which plots are alike but the surfaces of the area bounded by each of them is completely different. Both functions have the black holes at $+\infty$.

$$f_1: \mathbb{R}_+^* \to \mathbb{R} \;\; , \;\; f_1(x) = 2x \cdot \left[\frac{1}{x^2}\right]$$

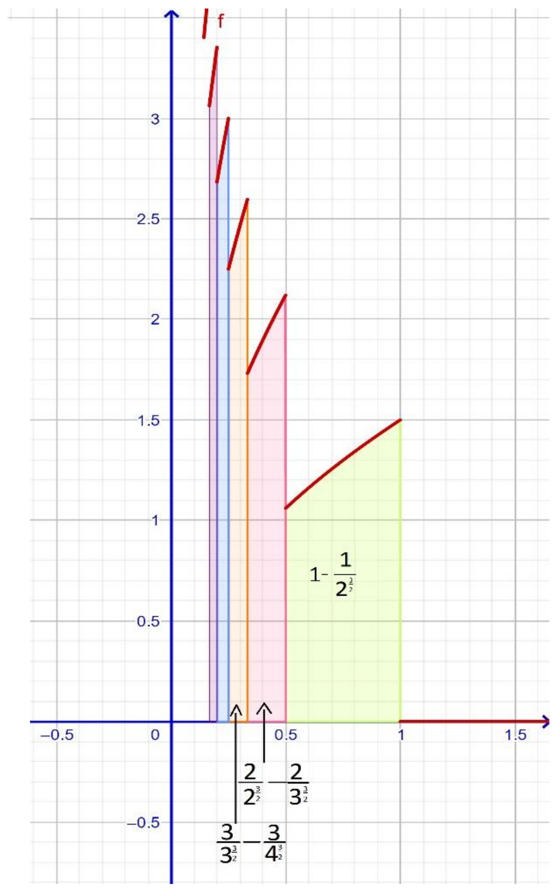
.

and

$$f_2: \mathbb{R}_+^* \to \mathbb{R} \;\; , \;\; f_2(x) = \frac{3}{2}\sqrt{x} \cdot \left[\frac{1}{x}\right]$$

We can already notice that the f_2 function is actually $g_{\frac{3}{2}}$ resulting that surface of its hypograph is equal to $\zeta(\frac{3}{2}) = 1 + \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \ldots \approx 2.61$

This mathematical series has multiple applications in different domains, for example: it is employed in calculating the critical temperature for Bose-Einstein condensate in a box with periodic boundary conditions, and for spin wave physic in magnetic systems.
Now, let us take another step further and calculate the area of the subgraph of the f_1.



We consider $x \in \left(\frac{1}{\sqrt{k+1}}, \frac{1}{\sqrt{k}}\right]$ resulting that $\left[\frac{1}{x^2}\right] = k$.

$$\int_0^1 f_1(x)dx = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \int_{\frac{1}{\sqrt{k+1}}}^{\frac{1}{\sqrt{k}}} 2kx\, dx\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} kx^2\Big|_{\frac{1}{\sqrt{k+1}}}^{\frac{1}{\sqrt{k}}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1-\frac{k}{k+1}\right)\right)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1=\frac{k+1-1}{k+1}\right)\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1-1+\frac{1}{k+1}\right)\right) = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) = \infty$$

Remark 1 Summing up the areas of hypograph of the f_2 on two consecutive intervals $(\frac{1}{k+2}, \frac{1}{k+1}]$ and $(\frac{1}{k+1}, \frac{1}{k}]$, we obtain two consecutive terms of $\zeta(\frac{3}{2})$.

Remark 2 Also the function $g(x) = 3x^2 \cdot \left[\frac{1}{x^2}\right]$ will generate $\zeta(\frac{3}{2})$.

As you can notice, the area of the hypograph of f_1 is $\zeta(1)$ which represents the Harmonic Series. This series is the source of many paradoxes, one of them is called „the worm on the rubber band". Suppose that a worm crawls along an infinitely-elastic one-meter rubber band at the same time as the rubber band is uniformly stretched. If the worm travels 1 centimeter per minute and the band stretches 1 meter per minute, will the worm ever reach the end of the rubber band? Although the answer appears to be „no", contrary to our intuition it is an affirmative one, for after n minutes, the ratio of the distance travelled by the worm to the total length of the rubber band is $\frac{1}{100}\sum_{k=1}^{n}\frac{1}{k}$. Because the series gets arbitrarily large as n becomes larger, eventually this ratio must exceed 1, which implies that the worm reaches the end of the rubber band. However, the value of n at which this occurs must be extremely large: approximately e^{100}, a number exceeding 10^{43} minutes (10^{37} years). Although the harmonic series does diverge, it does so very slowly.

3.2 Different functions generating the same series

We define:

$$f_3: \mathbb{R}^* \to \mathbb{R},\ f_3(x) = 4x^3 \cdot \left[\frac{1}{x}\right] \, and \, f_4: \mathbb{R}_+^* \to \mathbb{R}, f_4(x) = 2x \cdot \left[\frac{1}{\sqrt{x}}\right]$$

Both functions have the black hole in G(0,0). The plots are slightly different, the first one is constituted of an infinite number of curves whereas the second one is made up of an infinity of segments.

f_3 can be identified as g_4 function and the area of its hypograph generates $\zeta(4) = \frac{\pi^4}{90} \approx 1.0823$.

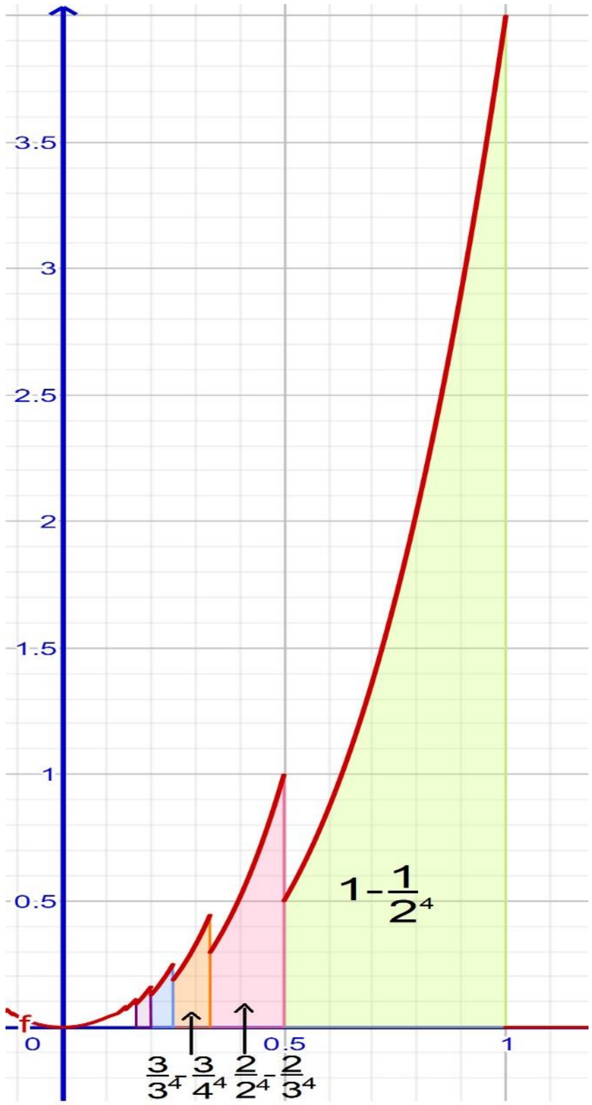
In order to calculate the area of the subgraph of f_4 we will consider:

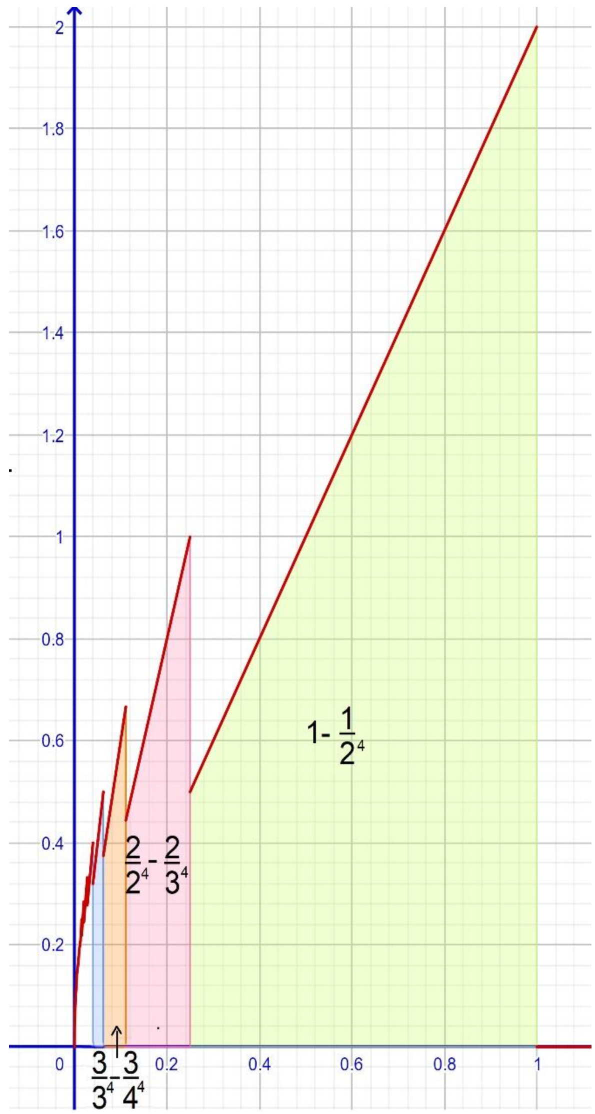
$$x \in \left(\frac{1}{(k+1)^2}, \frac{1}{k^2}\right], \qquad k \in \mathbb{N}^*$$

$$\int_0^1 f_4(x)dx = \lim_{n\to\infty}(\sum_{k=1}^{n}\int_{\frac{1}{(k+1)^2}}^{\frac{1}{k^2}} 2kxdx = \lim_{n\to\infty}(\sum_{k=1}^{n} kx^2|_{\frac{1}{(k+1)^2}}^{\frac{1}{k^2}} =$$

$$= \lim_{n\to\infty}(\sum_{k=1}^{n}(\frac{1}{k^3} - \frac{k}{(k+1)^4}) = \lim_{n\to\infty}(\sum_{k=1}^{n}(\frac{1}{k^3} - \frac{k+1-1}{(k+1)^4}) = \lim_{n\to\infty}((\sum_{k=1}^{n}\frac{1}{k^3} -$$

$$-\sum_{k=1}^{n}\frac{1}{(k+1)^3} + \sum_{k=1}^{n}\frac{1}{(k+1)^4}) = \lim_{n\to\infty}((1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{n^4}) = \zeta(4) = \frac{\pi^4}{90}$$



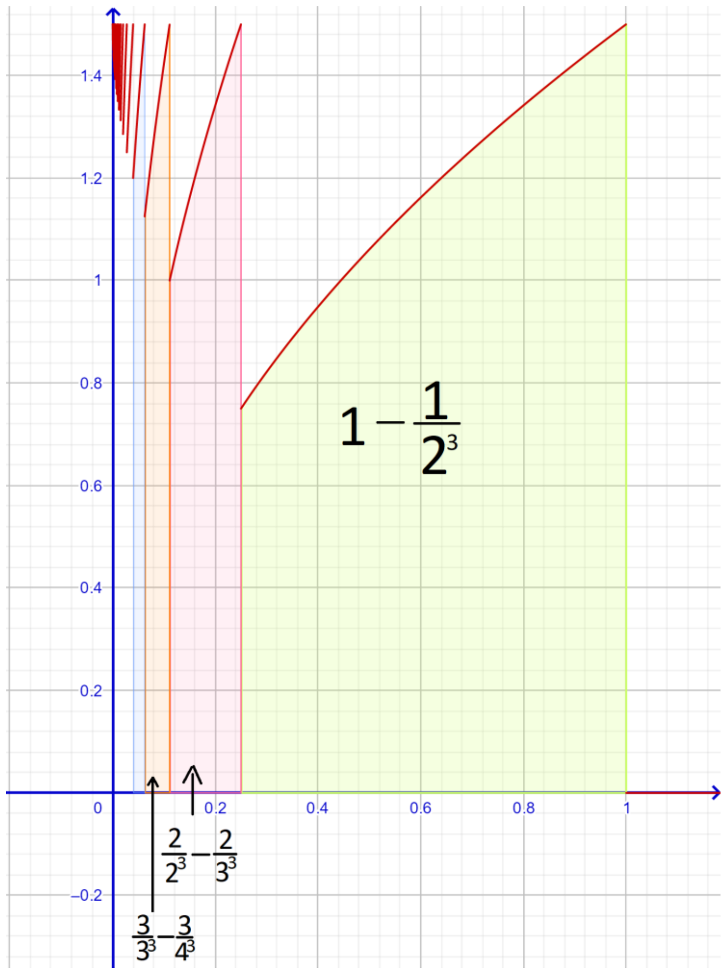


In this case as well, summing up the areas of the hypograph on two consecutive intervals generates two consecutive terms of the $\zeta(4)$ series.

3.3 A function whose area of the hypograph generates the famous $\zeta(3)$ series

In the end, we want to show you another „black hole" function. We will find an approximative value of the area of its hypograph and we propose you to dig deeper and find the exact value. This function is:

$$f_5 : \mathbb{R}_+^* \to \mathbb{R}, \qquad f_5(x) = \frac{3}{2}\sqrt{x} \cdot \left[\frac{1}{\sqrt{x}}\right]$$



To calculate the area of its hypograph, we consider

$$x \in \left(\frac{1}{(k+1)^2}, \frac{1}{k^2}\right], k \in \mathbb{N}^*$$

$$\int_0^1 f_5(x)dx = \lim_{n\to\infty}\left(\sum_{k=1}^n \int_{\frac{1}{(k+1)^2}}^{\frac{1}{k^2}} \frac{3}{2}k\sqrt{x} \quad dx = \lim_{n\to\infty}\left(\sum_{k=1}^n kx^{\frac{3}{2}}\Big|_{\frac{1}{(k+1)^2}}^{\frac{1}{k^2}} = \lim_{n\to\infty}\left(\sum_{k=1}^n \left(\frac{1}{k^2} - \right.\right.\right.\right.$$

$$- \frac{k}{(k+1)^3}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^n \left(\frac{1}{k^2} - \frac{k+1-1}{(k+1)^3}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k^2} - \sum_{k=1}^n \frac{1}{(k+1)^2} + \right.$$

$$+ \sum_{k=1}^n \frac{1}{(k+1)^3}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{(n+1)^3}\right) = \zeta(3).$$

At this moment, no one succeeded in calculating this mathematical series so we only know a approximative value: $\zeta(3) \approx 1.20205$.

$\zeta(3)$ is also known as Apéry's constant. This constant arises naturally in a number of physical problems. As it is an irrational number, it has an infinite number of decimal digits. The number of known decimal digits of Apéry's constant $\zeta(3)$ has increased dramatically during the last decades. This is due both to the increasing performance of computers and to algorithmic improvements. Leonhard Euler was the first to find a decimal for this number in 1735 and he kept on going until he found 16 decimals. The last one who found more decimals was Dipanjan Nag who managed to find no less than 400 billions decimals digits.

We give you the opportunity to be the first to calculate this series, as it is still considered to be one of the greatest unresolved mathematical problems.

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MP40. SOME INFINITE PRODUCTS

Dan Arteni, Tudor Podina
Romania, Satu Mare, Romania

ABSTRACT

The concept of an infinite product (or sum) is mysterious and intriguing. How can you multiply an infinite number of terms? And what is the result? In this paper we obtain some results using the monotony of some functions. Then we show the relation between the result and the terms of the product. In the end, we will calculate the sum of some series using the results of infinite products.

1. INTRODUCTION

Definition. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $p_n = \prod_{i=1}^{n} a_i$. The pair $(a_n)_{n\geq 1}$ and $(p_n)_{n\geq 1}$ is called an infinite product of real numbers, also written as $\prod_{n\geq 1} a_n$.
If $(p_n)_{n\geq 1}$ converges to a number $a \neq 1$, then the infinte product $\prod_{n\geq 1} a_n$ is convergent and it is written as $a = \prod_{n\geq 1} a_n$. The elements of $(a_n)_{n\geq 1}$ are called the terms of the product, whereas the elements of $(p_n)_{n\geq 1}$ are called partial products. An infinite product that is not convergent is called divergent. If $\lim_{n\to\infty} p_n = \infty$, then $\prod_{n\geq 1} a_n = \infty$. Analogously, if $\lim_{n\to\infty} p_n = -\infty$, then $\prod_{n\geq 1} a_n = -\infty$

Theorem 1. If $\prod_{n\geq 1} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 1$.

Proof.

$$a_{n+1} = \frac{p_{n+1}}{p_n}$$

taking the limit of the equality, it results into

$$\lim_{n\to\infty} a_{n+1} = \frac{\lim_{n\to\infty} p_{n+1}}{\lim_{n\to\infty} p_n} = \frac{p}{p} = 1.$$

Corollary. Let $\prod_{n\geq 1} (1 + a_n)$ *converge with* $a_n \in \mathbb{R}, \forall n \geq 1$. *Then* $a_n \neq 1$ *and* $\lim_{n\to\infty} a_n = 0$.

Theorem 2. The infinite product $\prod_{n\geq 1} a_n$ *is convergent* $\Rightarrow$ *the infinite product* $\prod_{n\geq 1} (a_n^{-1})$ *converges, then:*

$$\prod_{n\geq 1} (a_n^{-1}) = \left(\prod_{n\geq 1} a_n\right)^{-1}.$$

Examples.

1. The infinite product $\prod_{n\geq 1}(1+n)=\infty$.
Truly, $p_n=\prod_{i=1}^{n}(1+i)>1+n$, *therefore* $\lim_{n\to\infty} p_n=\infty$.

2. $\prod_{n\geq 2}\left(1+\frac{(-1)^n}{n}\right)=1$.

Let $p_{n-1}=\prod_{k=1}^{n}\left(1+\frac{(-1)^k}{k}\right)=\left(1+\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\cdot\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{(-1)^n}{n}\right)=$

$=\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{5}{4}\ldots\frac{n+(-1)^n}{n}=\begin{cases}1+\frac{1}{n}, n-even\\ 1\quad, n-odd\end{cases}$.

So, $\lim_{n\to\infty} p_n=1$.

3. $\prod_{n\geq 1}\frac{n(n+2)}{(n+1)^2}=\frac{1}{2}$.

Truly, $p_n=\prod_{k=1}^{n}\frac{k(k+2)}{(k+1)^2}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\ldots\frac{(n-1)(n+1)}{n\cdot n}\cdot\cdot\frac{n(n+2)}{(n+1)(n+1)}=\frac{n+2}{2n+1}$. *So,* $\lim_{n\to\infty} p_n=\frac{1}{2}$.

Theorem 3. Let $(a_n)_{n\geq 1}$ a sequence of strictly positive numbers. Then $\prod_{n\geq 1} a_n$ converges $\Longrightarrow$ the series $\sum_{n\geq 1}\ln a_n$ is convergent.

Proof. Let p_n and s_n be the partial product and the partial sum of the n order. This means that:

$$p_n=\prod_{k=1}^{n} a_k=e^{\prod_{k=1}^{n} ln a_k}=e^{s_n}, \forall n\geq 1$$

Therefore, the series $\sum_{n\geq 1}\ln a_n$ is convergent $\Longrightarrow \prod_{n\geq 1} a_n$ converges.

Theorem 4. Let $(a_n)_{n\geq 1}\subseteq\mathbb{R}\backslash 1$ *such that* $\prod_{n\geq 1} a_n^2<\infty$. *As a result,* $\prod_{n\geq 1}(1+a_n)$ *is convergent* $\Longrightarrow \sum_{n\geq 1} a_n$ *is convergent.*

Proof. As $\lim_{n\to 0}\frac{ln(1+x)-x}{x^2}=-\frac{1}{2}$ *it means that* $\forall[a,b]\subset(-1,\infty),\exists M>0$ *such that* $|ln(1+x)-x|\leq Mx^2,\forall x\in[a,b]$. *If the infinite product* $\prod_{n\geq 1}(1+a_n)$ *converges or its series* $\sum_{n\geq 1} a_n$ *is convergent, it is fair to assume that* $\exists(a_n)_{n\geq 1}$ *such that* $-1<a<b<\infty$ *with* $a_n\in[a,b],\forall n\geq 1$. *Then* $\exists M>0$ *such as*

$$|ln(1+a_n)-a_n|\leq Ma_n^2,\forall n\geq 1$$

so $\sum_{n\geq 1}|ln(1+a_n)-a_n|\leq\sum_{n\geq 1} a_n^2<\infty$. *From this it is concluded that the series* $\sum_{n\geq 1}(ln(1+a_n)-a_n)$ *is absolutely convergent, so the series* $\sum_{n\geq 1} a_n$ *and* $\sum_{n\geq 1} ln(1+a_n)$ *are of the same nature.*

However, the latter is of the same nature as the infinite product $\prod_{n\geq 1}(1+a_n)$, *so the series* $\sum_{n\geq 1} a_n$ *and the product* $\prod_{n\geq 1}(1+a_n)$ *are of the same nature.*

Theorem 5. Let $(a_n)_{n\geq 1}\subset\mathbb{R}_+^*$ or $(a_n)_{n\geq 1}\subset\mathbb{R}_-^*\backslash 1$. Then $\prod_{n\geq 1}(1+a_n)$ converges $\Longrightarrow \sum_{n\geq 1} a_n$ converges.

Proof. If the product $\prod_{n\geq 1}(1+a_n)$ *or the series* $\sum_{n\geq 1} a_n$ *is convergent, then* $a_n\to 0$,*so it is fair to assume that (with the exception of a finite number of terms)* $(a_n)_{n\geq 1}\subset(0,1)$ *or* $(a_n)_{n\geq 1}\subset(-1,0)$.

$\underset{n\to\infty}{Lim}\frac{ln(1+a_n)}{a_n}=\lim_{n\to\infty}\frac{-ln(1+a_n)}{-a_n}=1$.

If $(a_n)_{n\geq 1}\leq(0,1)$ *then the series* $\sum_{n\geq 1} a_n$ *and* $\sum_{n\geq 1} ln(1+a_n)$ *have strictly positive terms and are of the same nature.*

If $(a_n)_{n\geq 1} \leq (-1,0)$, rationalizing analogously, the series $\sum_{n\geq 1} -a_n$ is of the same nature as $\sum_{n\geq 1} -ln(1+a_n)$, so $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} ln(1+a_n)$ are of the same nature. Corollary1. The product $\prod_{n\geq 1}(1+a_n)$ is absolutely convergent $\Longrightarrow$ the series $\sum_{n\geq 1} a_n$ is absolutely convergent.

Corollary 2. The infinite products $\prod_{n\geq 1} 1+\frac{1}{n^\alpha}$ and $\prod_{n\geq 1} 1-\frac{1}{n^\alpha}$, $\forall\alpha\in\mathbb{R}$ converge for $\alpha>1$ and diverge for $\alpha\leq 1$.

Observation. In the study of convergent infinite products of real numbers it is sufficient to consider products like $\prod_{n\geq 1} a_n$ with $a_n>0, \forall n\geq 1$ or $\prod_{n\geq 1}(1+a_n)$ with $a_n>0, \forall n\geq 1$.

2. SOME INFINITE PRODUCTS

Now we will calculate some infinite products using the monotony of adequately chosen functions. We intent to calculate the following products:

$$(P_1)\prod_{n\geq 1}\frac{4n}{4n+1};\qquad (P_2)\prod_{n\geq 1}\frac{4n}{4n+2};\qquad (P_3)\prod_{n\geq 1}\frac{4n}{4n+3}$$

We will begin with P_1. For that matter, let us consider the function

$f:(1,\infty)\to\mathbb{R}, f(x)=\frac{x}{x+1}$. We get that $f'(x)=\frac{x+1-x}{(x+1)^2}=\frac{1}{(x+1)^2}>0$ which means that f is strictly increasing. For $x_1=4k$ and $x_2=4k+1$ we have that $x_1<x_2$, meaning that $f(x_1)<f(x_2)$ which is equivalent to $\frac{4k}{4k+1}<\frac{4k+1}{4k+2}$, where $k\in\mathbb{N}^*$.

In order to calculate P_1 we will use the notation $\prod_{k=1}^{n}\frac{4k}{4k+1}=p_n$. Furthermore:

$$p_n=\frac{4}{5}\cdot\frac{8}{9}\cdot\frac{12}{13}\cdot\ldots\cdot\frac{4n}{4n+1}$$

As we intent to majorate, we will raise p_n to the fourth power in order to be able to find a superior margin. This is written as:

$$p_n^4=\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{12}{13}\cdot\frac{12}{13}\cdot\frac{12}{13}\cdot\frac{12}{13}\cdot\ldots\cdot\frac{4n}{4n+1}\cdot\frac{4n}{4n+1}\cdot\frac{4n}{4n+1}\cdot\frac{4n}{4n+1}$$

$$p_n^4<\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{7}{8}\cdot\frac{8}{9}\cdot\frac{9}{10}\cdot\frac{10}{11}\cdot\frac{11}{12}\cdot\frac{12}{13}\cdot\frac{13}{14}\cdot\frac{14}{15}\cdot\frac{15}{16}\cdot\ldots\cdot\frac{4n}{4n+1}\cdot\frac{4n+1}{4n+2}\cdot\frac{4n+2}{4n+3}\cdot\frac{4n+3}{4n+4}=$$

$$=\frac{4}{4n+4}=\frac{1}{n+1}$$

Because $0<p_n^4<\frac{1}{n+1}$ we have $0<p_n<\sqrt[4]{\frac{1}{n+1}}$. Taking it to the limit as n goes to ∞ we get that $\lim_{n\to\infty}p_n=0$. In other words, $\prod_{n\geq 1}\frac{4n}{4n+1}=0$.

We will now proceed towards the calculation of P_2. In this case, we use the function f☹1,$\infty)\to\mathbb{R}, f(x)=\frac{x}{x+2}$. Analogously, f is stricly increasing. Thus, for $x_1=4k$ and $x_2=4k+1$ we get that $f(x_1)<f(x_2)$ which is written as $\frac{4k}{4k+2}<\frac{4k+1}{4k+3}$, where $k\in\mathbb{N}^*$.

We will now use the notation $\prod_{k=1}^{n}\frac{4k}{4k+2}=p_n$. *In other words:*
$p_n=\frac{4}{6}\cdot\frac{8}{10}\cdot\frac{12}{14}\cdot\ldots\cdot\frac{4n}{4n+2}$

Raising again to the fourth power, we have:

$$p_n^4=\frac{4}{6}\cdot\frac{4}{6}\cdot\frac{4}{6}\cdot\frac{4}{6}\cdot\frac{8}{10}\cdot\frac{8}{10}\cdot\frac{8}{10}\cdot\frac{8}{10}\cdot\frac{12}{14}\cdot\frac{12}{14}\cdot\frac{12}{14}\cdot\frac{12}{14}\cdot\ldots\cdot\frac{4n}{4n+2}\cdot\frac{4n}{4n+2}\cdot\frac{4n}{4n+2}\cdot\frac{4n}{4n+2}$$

$$p_n^4<\frac{4}{6}\cdot\frac{5}{7}\cdot\frac{6}{8}\cdot\frac{7}{9}\cdot\frac{8}{10}\cdot\frac{9}{11}\cdot\frac{10}{12}\cdot\frac{11}{13}\cdot\frac{12}{14}\cdot\frac{13}{15}\cdot\frac{14}{16}\cdot\frac{15}{17}\cdot\ldots\cdot\frac{4n}{4n+2}\cdot\frac{4n+1}{4n+3}\cdot\frac{4n+2}{4n+4}\cdot\frac{4n+3}{4n+5}$$

$$=\frac{4\cdot5}{(4n+4)(4n+5)}$$

We are yet again left with $0<p_n^4<\frac{4\cdot5}{(4n+4)(4n+5)}$ *which can be written as* $0<p_n<\sqrt[4]{\frac{4\cdot5}{(4n+4)(4n+5)}}$. *Taking it to the limit we have* $\lim_{n\to\infty}p_n=0$, *therefore* $\prod_{n\geq1}\frac{4n}{4n+2}=0$.

It is now the time to calculate P_3. We will proceed as before.

Let $f:[1,\infty)\to\mathbb{R}, f(x)=\frac{x}{x+3}$. *As* $f'(x)>0, f$ *is strictly increasing. Moreover, for* $x_1=4k$ *and* $x_2=4k+1$ *we get that* $x_1<x_2$, *which means that* $f(x_1)<f(x_2)$ *and that* $\frac{4k}{4k+3}<\frac{4k+1}{4k+4}$. *In addition, we will use the notation* $p_n=\prod_{k=1}^{n}\frac{4k}{4k+3}=\frac{4}{7}\cdot\frac{8}{11}\cdot\frac{12}{15}\cdot\ldots\cdot\frac{4n}{4n+3}$. *As a consequence of raising* p_n *to the fourth power, we have:*

$$p_n^4=\frac{4}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}\cdot\frac{8}{11}\cdot\frac{8}{11}\cdot\frac{8}{11}\cdot\frac{8}{11}\cdot\frac{12}{15}\cdot\frac{12}{15}\cdot\frac{12}{15}\cdot\frac{12}{15}\cdot\ldots\cdot\frac{4n}{4n+3}\cdot\frac{4n}{4n+3}\cdot\frac{4n}{4n+3}\cdot\frac{4n}{4n+3}$$

$$p_n^4<\frac{4}{7}\cdot\frac{5}{8}\cdot\frac{6}{9}\cdot\frac{7}{10}\cdot\frac{8}{11}\cdot\frac{9}{12}\cdot\frac{10}{13}\cdot\frac{11}{14}\cdot\frac{12}{15}\cdot\frac{13}{16}\cdot\frac{14}{17}\cdot\frac{15}{18}\cdot\ldots\cdot\frac{4n}{4n+3}\cdot\frac{4n+1}{4n+4}\cdot\frac{4n+2}{4n+5}\cdot\frac{4n+3}{4n+6}=$$

$$=\frac{4\cdot5\cdot6}{(4n+4)(4n+5)(4n+6)}$$

Finally, $0<p_n^4<\frac{4\cdot5\cdot6}{(4n+4)(4n+5)(4n+6)}$ *which is written as*

$$0<p_n<\sqrt[4]{\frac{4\cdot5\cdot6}{(4n+4)(4n+5)(4n+6)}}.$$ *Therefore,* $\lim_{n\to\infty}p_n=0$ *meaning that* $\prod_{n\geq1}\frac{4n}{4n+3}=0$.

Taking into consideration the finite sequence between the n^{th} and $2n^{th}$ term, no matter how large the value of a chosen n is, we get the following infinite products:

$$(L_1)\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{4k}{4k+1};\qquad (L_2)\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{4k}{4k+2};\qquad (L_3)\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{4k}{4k+3}.$$

We will now proceed with the calculation of L_1. Much like before, we apply the squeeze theorem on the number $p_n = \prod_{k=n}^{2n} \frac{4k}{4k+1}$ raised to the fourth power. Consequently, we have that, because

$$p_n = \frac{4n}{4n+1} \cdot \frac{4n+4}{4n+5} \cdot \frac{4n+8}{4n+9} \cdot \ldots \cdot \frac{8n}{8n+1}$$

when we raise it to the power, we have:

$$p_n^4 = \frac{4n}{4n+1} \cdot \frac{4n}{4n+1} \cdot \frac{4n}{4n+1} \cdot \frac{4n}{4n+1} \cdot \frac{4n+4}{4n+5} \cdot \ldots \cdot \frac{8n}{8n+1} \cdot \frac{8n}{8n+1} \cdot \frac{8n}{8n+1} \cdot \frac{8n}{8n+1}$$

$$p_n^4 < \frac{4n}{4n+1} \cdot \frac{4n+1}{4n+2} \cdot \frac{4n+2}{4n+3} \cdot \frac{4n+3}{4n+4} \cdot \frac{4n+4}{4n+5} \cdot \ldots \cdot \frac{8n}{8n+1} \cdot \frac{8n+1}{8n+2} \cdot \frac{8n+2}{8n+3} \cdot \frac{8n+3}{8n+4} =$$

$$= \frac{4n}{8n+4} = \frac{n}{2n+1}$$

$$p_n^4 > \frac{4n-3}{4n-2} \cdot \frac{4n-2}{4n-1} \cdot \frac{4n-1}{4n} \cdot \frac{4n}{4n+1} \cdot \ldots \cdot \frac{8n-3}{8n-2} \cdot \frac{8n-2}{8n-1} \cdot \frac{8n-1}{8n} \cdot \frac{8n}{8n+1} = \frac{4n-3}{8n+1}$$

We can now write the following inequality:

$$\frac{4n-3}{8n+1} < p_n^4 < \frac{n}{2n+1}$$

out of which, as a result of taking to the limit when n goes towards ∞, we get $\lim_{n\to\infty} p_n = \sqrt[4]{\frac{1}{2}}$ meaning that

$$\lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+1} = \sqrt[4]{\frac{1}{2}}$$

Now, we will calculate L_2. Proceeding similarly with P_2 we have:

$$p_n = \frac{4n}{4n+2} \cdot \frac{4n+4}{4n+6} \cdot \frac{4n+8}{4n+10} \cdot \ldots \cdot \frac{8n}{8n+2}$$

Which, raised to the fourth power, is:

$$p_n^4 = \frac{4n}{4n+2} \cdot \frac{4n}{4n+2} \cdot \frac{4n}{4n+2} \cdot \frac{4n}{4n+2} \cdot \frac{4n+4}{4n+6} \cdot \ldots \cdot \frac{8n}{8n+2} \cdot \frac{8n}{8n+2} \cdot \frac{8n}{8n+2} \cdot \frac{8n}{8n+2}$$

$$p_n^4 < \frac{4n}{4n+2} \cdot \frac{4n+1}{4n+3} \cdot \frac{4n+2}{4n+4} \cdot \frac{4n+3}{4n+5} \cdot \frac{4n+4}{4n+6} \cdot \ldots \cdot \frac{8n}{8n+2} \cdot \frac{8n+1}{8n+3} \cdot \frac{8n+2}{8n+4} \cdot \frac{8n+3}{8n+5} =$$

$$= \frac{4n(4n+1)}{(8n+4)(8n+5)}$$

$$p_n^4 > \frac{4n-3}{4n-1} \cdot \frac{4n-2}{4n} \cdot \frac{4n-1}{4n+1} \cdot \frac{4n}{4n+2} \cdot \ldots \cdot \frac{8n-2}{8n} \cdot \frac{8n-1}{8n+1} \cdot \frac{8n}{8n+2} = \frac{(4n-3)(4n-2)}{(8n+1)(8n+2)}$$

As a consequence, we get the inequality

$$\frac{4n(4n+1)}{(8n+4)(8n+5)} < p_n^4 < \frac{(4n-3)(4n-2)}{(8n+1)(8n+2)}$$

which, taken to the limit translates into $\lim_{n\to\infty} p_n = \sqrt[4]{\frac{1}{4}}$ meaning

$$\lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+2} = \sqrt[4]{\frac{1}{4}}$$

In the case of L_3 we have

$$p_n = \frac{4n}{4n+3} \cdot \frac{4n+4}{4n+7} \cdot \frac{4n+8}{4n+11} \cdot \ldots \cdot \frac{8n}{8n+3}$$

Raising to the fourth power we get

$$p_n^4 = \frac{4n}{4n+3} \cdot \frac{4n}{4n+3} \cdot \frac{4n}{4n+3} \cdot \frac{4n}{4n+3} \cdot \frac{4n+4}{4n+7} \cdot \ldots \cdot \frac{8n}{8n+3} \cdot \frac{8n}{8n+3} \cdot \frac{8n}{8n+3} \cdot \frac{8n}{8n+3}$$

$$p_n^4 < \frac{4n}{4n+3} \cdot \frac{4n+1}{4n+4} \cdot \frac{4n+2}{4n+5} \cdot \frac{4n+3}{4n+6} \cdot \frac{4n+4}{4n+7} \cdot \ldots \cdot \frac{8n}{8n+3} \cdot \frac{8n+1}{8n+4} \cdot \frac{8n+2}{8n+5} \cdot \frac{8n+3}{8n+6} =$$

$$= \frac{4n(4n+1)(4n+2)}{(8n+4)(8n+5)(8n+6)}$$

$$p_n^4 > \frac{4n-3}{4n} \cdot \frac{4n-2}{4n+1} \cdot \frac{4n-1}{4n+2} \cdot \frac{4n}{4n+3} \cdot \ldots \cdot \frac{8n-2}{8n+1} \cdot \frac{8n-1}{8n+2} \cdot \frac{8n}{8n+3} =$$

$$= \frac{(4n-3)(4n-2)(4n-1)}{(8n+1)(8n+2)(8n+3)}$$

Writing the inequality

$$\frac{4n(4n+1)(4n+2)}{(8n+4)(8n+5)(8n+6)} < p_n^4 < \frac{(4n-3)(4n-2)(4n-1)}{(8n+1)(8n+2)(8n+3)}$$

we can observe that $\lim_{n\to\infty} p_n = \sqrt[4]{\frac{1}{8}}$, meaning

$$\lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+3} = \sqrt[4]{\frac{1}{8}}.$$

3. GENERALIZATION

From our previous examples we can derive the formula $\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{4k}{4k+b}=\sqrt[4]{\frac{1}{2^b}}$.

Proof.

$$p_n=\frac{4n}{4n+b}\cdot\frac{4n+4}{4n+b+4}\dots\frac{8n}{8n+b}$$

Raising to the fourth:

$$p_n^4=\frac{4n}{4n+b}\cdot\frac{4n}{4n+b}\cdot\frac{4n}{4n+b}\cdot\frac{4n}{4n+b}\cdot\frac{4n+4}{4n+b+4}\dots\frac{8n}{8n+b}\cdot\frac{8n}{8n+b}\cdot\frac{8n}{8n+b}\cdot\frac{8n}{8n+b}$$

$$p_n^4<\frac{4n}{4n+b}\cdot\frac{4n+1}{4n+b+1}\dots\frac{4n+b-1}{4n+2b-1}\cdot\frac{4n+b}{4n+2b}\dots\frac{8n+2}{8n+b+2}\cdot\frac{8n+3}{8n+b+3}=$$

$$=\frac{4n(4n+1)(4n+2)\dots(4n+b-1)}{(8n+4)(8n+5)\dots(8n+b+3)}$$

$$p_n^4>\frac{4n-b}{4n}\cdot\frac{4n-b+1}{4n+1}\cdot\frac{4n-b+2}{4n+2}\dots\frac{4n}{4n+b}\cdot\frac{4n+1}{4n+b+1}\dots\frac{8n-1}{8n+b-1}\cdot\frac{8n}{8n+b}=$$

$$=\frac{(4n-b)(4n-b+1)\dots(4n-1)}{(8n+1)(8n+2)\dots(8n+b)}$$

Having the inequality

$$\frac{(4n-b)(4n-b+1)\dots(4n-1)}{(8n+1)(8n+2)\dots(8n+b)}<p_n^4<\frac{4n(4n+1)\dots(4n+b-1)}{(8n+4)(8n+5)\dots(8n+b+3)}$$

taking the limit we get that $\lim_{n\to\infty}p_n=\sqrt[4]{\frac{1}{2^b}}$, meaning $\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{4k}{4k+b}=\sqrt[4]{\frac{1}{2^b}}$.

Another parameter of the product that can be generalized is the coefficient of n. Now, we are able to write the product as $p_n=\prod_{k=n}^{2n}\frac{ak}{ak+b}$ and assume that $\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{ak}{ak+b}=\sqrt[a]{\frac{1}{2^b}}$.

Proof.

$$p_n=\frac{an}{an+b}\cdot\frac{an+a}{an+b+a}\cdot\frac{an+2a}{an+b+2a}\dots\frac{2an}{2an+b}$$

In order to simplify some terms of the product we need to raise to the a^{th} power:

$$p_n^a=\frac{an}{an+b}\cdot\frac{an}{an+b}\dots\frac{an}{an+b}\dots\frac{2an}{2an+b}\dots\frac{2an}{2an+b}$$

$$p_n^a<\frac{an}{an+b}\cdot\frac{an+1}{an+b+1}\dots\frac{an+b}{an+2b}\dots\frac{2an+a-2}{2an+b+a-2}\cdot\frac{2an+a-1}{2an+b+a-1}=$$

$$=\frac{an(an+1)(an+2)\dots(an+b-1)}{(2an+a)(2an+a+1)\dots(2an+b+a-1)}$$

$$p_n^a>\frac{an-b}{an}\cdot\frac{an-b+1}{an+1}\dots\frac{an}{an+b}\dots\frac{2an-1}{2an+b-1}\cdot\frac{2an}{2an+b}=\frac{(an-b)(an-b+1)\dots(an-1)}{(2an+1)(2an+2)\dots(2an+b)}$$

Writing the inequality and taking the limit we arrive at

$$\lim_{n\to\infty}\prod_{k=n}^{2n}\frac{ak}{ak+b}=\sqrt[a]{\frac{1}{2^b}}.$$

4. SOME INFINITE SUMS

An infinite sum that seems interesting to calculate is similar to the product presented at L_1. This is written as (S_1): $\sum_{k=n}^{2n} \ln\frac{4k}{4k+1}$. At a first sight, it appears to be difficult to calculate. Indeed, a direct approach seems impossible to spot. Therefore, we propose the following method:

$$\lim_{n\to\infty} \sum_{k=n}^{2n} ln\frac{4k}{4k+1} = \lim_{n\to\infty} ln \prod_{k=n}^{2n} \frac{4k}{4k+1} = ln \lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+1} = ln\sqrt[4]{\frac{1}{2}}.$$

Another interesting example derives from L2:

$$(S_2): \lim_{n\to\infty} \sum_{k=n}^{2n} ln\frac{4k}{4k+2}$$

Let us consider $\prod_{k=n}^{2n} \frac{4k}{4k+2}$ and we have already proven at L_2 that $\lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+2} = \sqrt[4]{\frac{1}{4}}$. Applying the logarithm we get that $\ln \lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+2} = \ln\sqrt[4]{\frac{1}{4}}$. Or that $\ln \lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+2} = \frac{1}{4}\ln\frac{1}{4}$. From here we can say that $\lim_{n\to\infty} \sum_{k=n}^{2n} \ln\frac{4k}{4k+2} = \frac{1}{4}\ln\frac{1}{4}$.

$$(S_3): \lim_{n\to\infty} \sum_{k=n}^{2n} ln\frac{4k}{4k+3}$$

Let us consider $\prod_{k=n}^{2n} \frac{4k}{4k+3}$ and we have already proven at L_3 that $\lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+3} = \sqrt[4]{\frac{1}{8}}$. Applying the logarithm we get that $\ln \lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+3} = \ln\sqrt[4]{\frac{1}{8}}$. Or that $\ln \lim_{n\to\infty} \prod_{k=n}^{2n} \frac{4k}{4k+3} = \frac{1}{4}\ln\frac{1}{8}$ and that $\lim_{n\to\infty} \sum_{k=n}^{2n} \ln\frac{4k}{4k+3} = \frac{1}{4}\ln\frac{1}{8}$.

Analoguosly, many other infinite sums can be found using infinte products.

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MP41. MATHEMATIC APPLICATION IN MEDICAL SCIENCE

Mina Behroozi
Imam Reza, Iran

Medicine in math

Prescriptions and Medication

Regularly, doctors write prescriptions to their patients for various ailments. Prescriptions indicate a specific medication and dosage amount. Most medications have guidelines for dosage amounts in milligrams (mg) per kilogram (kg). Doctors need to figure out how many milligrams of medication each patient will need, depending on their weight. If the weight of a patient is only known in pounds, doctors need to convert that measurement to kilograms and then find the amount of milligrams for the prescription. There is a very big difference between mg/kg and mg/lbs, so it is imperative that doctors understand how to accurately convert weight measurements. Doctors must also determine how long a prescription will last. For example, if a patient needs to take their medication, say one pill, three times a day. Then one month of pills is approximately 90 pills. However, most patients prefer two or three month prescriptions for convenience and insurance purposes. Doctors must be able to do these calculations mentally with speed and accuracy.

Doctors must also consider how long the medicine will stay in the patient's body. This will determine how often the patient needs to take their medication in order to keep a sufficient amount of the medicine in the body. For example, a patient takes a pill in the morning that has 50mg of a particular medicine. When the patient wakes up the next day, their body has washed out 40% of the medication. This means that 20mg have been washed out and only 30mg remain in the body. The patient continues to take their 50mg pill each morning. This means that on the morning of day two, the patient has the 30mg left over from day one, as well as another 50mg from the morning of day two, which is a total of 80mg. As this continues, doctors must determine how often a patient needs to take their medication, and for how long, in order to keep enough medicine in the patient's body to work effectively, but without overdosing.

The amount of medicine in the body after taking a medication decreases by a certain percentage in a certain time (perhaps 10% each hour, for example). This percentage decrease can be expressed as a rational number, 1/10. Hence in each hour, if the amount at the end of the hour decreases by 1/10 then the amount remaining is 9/10 of the amount at the beginning of the hour. This constant rational decrease creates a geometric sequence. So, if a patient takes a pill that has 200mg of a certain drug, the decrease of medication in their body each hour can be seen in the folowing table. The **Start** column contains the number of mg of the drug remaining in the system at the start of the hour and the **End** column contains the number of mg of the drug remaining in the system at the end of the hour.

Hour	***Start***	***End***
1	*200*	*9/10 x 200 = 180*
2	*180*	*9/10 x 180 = 162*
3	*162*	*9/10 x 162 = 145.8*
.	*.*	*.*

The sequence of numbers shown above is geometric because there is a common ratio between terms, in this case 9/10. Doctors can use this idea to quickly decide how often a patient needs to take their prescribed medication.

Ratios and Proportions

Nurses also use ratios and proportions when administering medication. Nurses need to know how much medicine a patient needs depending on their weight. Nurses need to be able to understand the doctor's orders. Such an order may be given as: 25 mcg/kg/min. If the patient weighs 52kg, how many milligrams should the patient receive in one hour? In order to do this, nurses must convert micrograms (mcg) to milligrams (mg). If 1mcg = 0.001mg, we can find the amount (in mg) of 25mcg by setting up a proportion.

$$\frac{1}{0.001} = \frac{25}{x}$$

By cross-multiplying and dividing, we see that 25mcg = 0.025mg. If the patient weighs 52kg, then the patient receives 0.025(52) = 1.3mg per minute. There are 60 minutes in an hour, so in one hour the patient should receive 1.3(60) = 78mg. Nurses use ratios and proportions daily, as well as converting important units. They have special "shortcuts" they use to do this math accurately and efficiently in a short amount of time.

Numbers give doctors much information about a patient's condition. White blood cell counts are generally given as a numerical value between 4 and 10. However, a count of 7.2 actually means that there are 7200 white blood cells in each drop of blood (about a microlitre). In much the same way, the measure of creatinine (a measure of kidney function) in a blood sample is given as X mg per deciliter of blood. Doctors need to know that a measure of 1.3 could mean some extent of kidney failure. Numbers help doctors understand a patient's condition. They provide measurements of health, which can be warning signs of infection, illness, or disease.

Body Mass Index

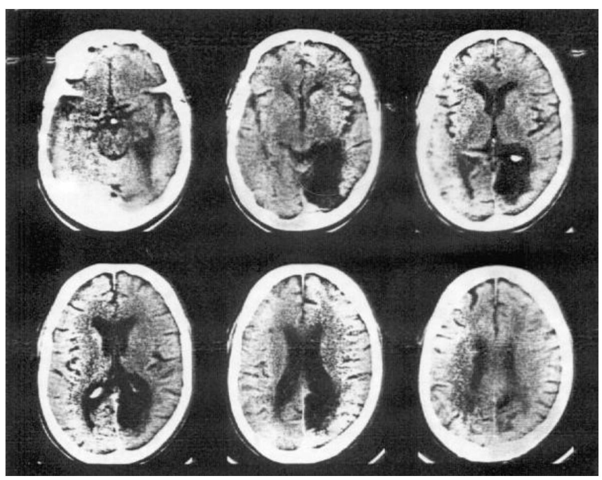
In terms of medicine and health, a person's Body Mass Index (BMI) is a useful measure. Your BMI is equal to your weight in pounds, times 704.7, divided by the square of your height in inches. This method is not always accurate for people with very high muscle mass because the weight of muscle is greater than the weight of fat. In this case, the calculated BMI measurement may be misleading. There are special machines that find a person's BMI. We can find the BMI of a 145-pound woman who is 5'6" tall as follows.

First, we need to convert the height measurement of 5'6" into inches, which is 66". Then, the woman's BMI would be:

$$\frac{145 \bullet 704.7}{66^2} = 23.4576$$

This is a normal Body Mass Index. A normal BMI is less than 25. A BMI between 25 and 29.9 is considered to be overweight and a BMI greater than 30 is considered to be obese. BMI measurements give doctors information about a patient's health. Doctor's can use this information to suggest health advice for patients. The image below is a BMI table that gives an approximation of health and unhealthy body mass indexes.

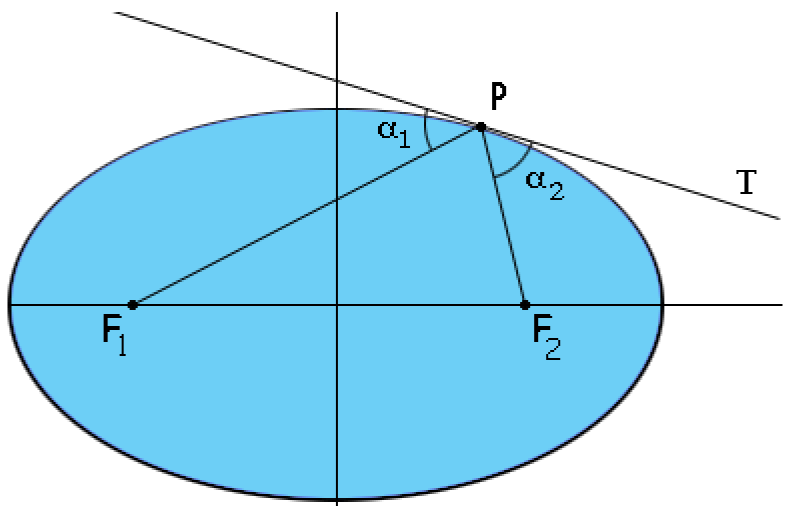
One of the more advanced ways that medical professionals use mathematics is in the use of CAT scans. A CAT scan is a special type of x-ray called a Computerized Axial Tomography Scan. A regular x-ray can only provide a two-dimensional view of a particular part of the body. Then, if a smaller bone is hidden between the x-ray machine and a larger bone, the smaller bone cannot be seen. It is like a shadow.



It is much more beneficial to see a three dimensional representation of the body's organs, particularly the brain. CAT scans allow doctors to see *inside*the brain, or another body organ, with a three dimensional image. In a CAT scan, the x-ray machine moves around the body scanning the brain (or whichever body part is being scanned) from hundreds of different angles. Then, a computer takes all the scans together and creates a three dimensional image. Each time the x-ray machine makes a full revolution around the brain, the machine is producing an image of a thin slice of the brain, starting at the top of the head and moving down toward the neck. The three-dimensional view created by the CAT scan provides much more information to doctors that a simple two-dimensional x-ray.
Mathematics plays a crucial role in medicine and because people's lives are involved, it is very important for nurses and doctors to be very accurate in their mathematical

calculations. Numbers provide information for doctors, nurses, and even patients. Numbers are a way of communicating information, which is very important in the medical field.
Another application of mathematics to medicine involves a lithotripter. This is a medical device that uses a property of an ellipse to treat gallstones and kidney stones.

The ellipse is a very special and practical conic section. One important property of the ellipse is its reflective property. If you think of an ellipse as being made from a reflective material then a light ray emitted from one focus will reflect off the ellipse and pass through the second focus. This is also true not only for light rays, but also for other forms of energy, including shockwaves. Shockwaves generated at one focus will reflect off the ellipse and pass through the second focus. This characteristic, unique to the ellipse, has inspired a useful medical application. Medical specialists have used the ellipse to create a device that effectively treats kidney stones and gallstones. A *lithotripter* uses shockwaves to successfully shatter a painful kidney stone (or gallstone) into tiny pieces that can be easily passed by the body. This process is known as lithotripsy.



As illustrated in the diagram above, when as energy ray reflects off a surface, the angle of incidence is equal to the angle of reflection.

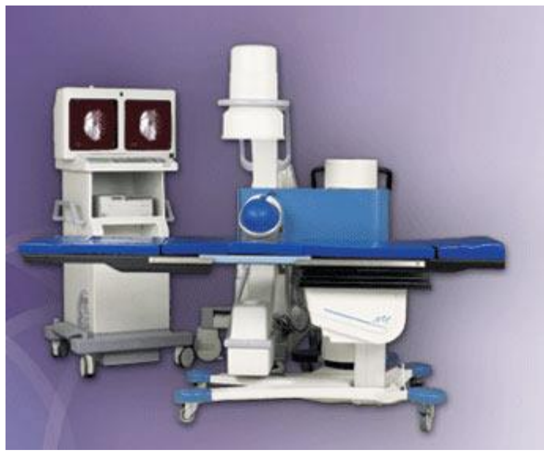
$$\alpha_1 = \alpha_2$$

Extracorporeal Shockwave Lithotripsy

Extracorporeal Shockwave Lithotripsy (ESWL) enables doctors to treat kidney and gall stones without open surgery. By using this alternative, risks associated with surgery are significantly reduced. There is a smaller possibility of infections and less recovery time is required than for a surgical procedure. The lithotripter is the instrument used in lithotripsy. The mathematical properties of an ellipse provide the basis for this medical invention.

Shockwaves

Electrohydraulic, piezoelectric, and electromagnetic energy systems use the focus of the ellipsoid to create the shockwaves needed to fracture the stone. The waves are generated at one focus and because of the elliptical shape, the waves are redirected onto the second focus, which is the stone itself. All of these waves cause the stone to crack and it eventually fragments into many tiny pieces that can then be easily passed by the body.



The process of lithotripsy takes about an hour. The patient can usually return home the same day and is not subjected to a lengthy recovery that is frequently required after surgery. Lithotripsy is virtually painless. The vibration and noise of the shocks can be uncomfortable and so most patients require minimal anesthesia. For these reasons, lithotripsy is becoming a popular treatment for many patients.

MP51. CONTINUED FRACTIONS

Ivayla Ivaylova Radkova
125SU "Boyan Penev"; Bulgaria

ABSTRACT

What is a continued fraction? Is it possible irrational number to equal a fraction? If we have a continued fraction how can we find its value? A historical reference. The problem with the calendar and the duration of the year. Representation of π as a continued fraction and the history of finding its first partial fractions. The infinite continued fraction which contains only the digit 1 (φ(the golden sequence)). Fibonacci sequence and fibonacci numbers. The most irrational number?! How to solve a quadratic equation using continued fractions? We are going to take a look at all these topics and answer the questions.

WHAT IS A CONTINUED FRACTION?

Actually the name speaks for itself. The constructed continued fraction looks like a chain made out of fractions. It's basically a fraction that we continue and probably that's where the name comes from. For precision we are going to separate the whole part from each fraction that we get and we are going to make the fractional part's numerator equal to 1. Thus we can generalize the look of the continued fractions like that:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Where $a_0 \in \mathbb{Z}_0^+,\ a_i \in \mathbb{N}$ ($i \in \mathbb{N}$). By generalizing the look of the continued fraction like that we achieve the uniqueness of the representation of each real number as a continued fraction (this can be proven with Euclid's algorithm).

The process (of constructing the continued fraction) may end or it may not. If it ends (if we have a finite continued fraction) the real number that we've represented like this finite continued fraction is rational and if it doesn't end (if we have an infinite continued fraction) the real number that we've represented like this infinite continued fraction is irrational.

IRRATIONAL NUMBER EQUALS A FRACTION?!

Well basically the infinite continued fraction is not a fraction. It doesn't have an end and that's why we can't represent it as a fraction. So we can all breathe a sigh of relief: irrational numbers exist.

IF WE HAVE A CONTINUED FRACTION HOW CAN WE FIND ITS VALUE?

We can find the value of any finite continued fraction just by starting from the “bottom” of the continued fraction. For example:

$$5+\cfrac{1}{8+\cfrac{1}{17+\cfrac{1}{13}}}=5+\cfrac{1}{8+\cfrac{1}{\frac{222}{13}}}=5+\cfrac{1}{8+\frac{13}{222}}=5+\cfrac{1}{\frac{1789}{222}}=5+\frac{222}{1789}=\frac{9167}{1789}$$

But the infinite continued fractions don’t have a ”bottom” so we can’t do the same thing as we did with the finite continued fractions. Because of that there are partial fractions of every infinite continued fraction. It is considered to mark them with K_i, where $i \in \mathbb{Z}_0^+$ and

$$K_i = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_i}}}$$

Because of Christiaan Huygens we know that the partial fractions with an even index are smaller than the number that we’ve represented like a continued fraction and the ones with an odd index are bigger than the number and that they (the partial fractions) are the best rational approximations to the number.

A LITTLE HISTORICAL REFERENCE

The first the continued fractions may be traced back to around 365 BC when Theon of Alexandria used them to find the approximate values of $\sqrt{a^2+r}$ (a, r ∈ ℕ). His method can be reduced to calculating the following continued fractions:

$$a\,;\,a+\frac{r}{2a}\,;\,a+\cfrac{r}{2a+\cfrac{r}{2a}}\,;\,a+\cfrac{r}{2a+\cfrac{r}{2a+\cfrac{r}{2a}}}\ \ldots$$

The finite continued fractions can be found in the period 306 – 283 year before Christ at the same time as Euclid's algorithm for calculating the greatest common divisor of two natural numbers.

Archimedes used them as approximate values of some irrational numbers. This way for approximate values of $3\sqrt{3}$ he suggests $\frac{265}{51}=5+\frac{1}{5+\frac{1}{10}}$ and $\frac{1351}{260}=5+\frac{1}{5+\frac{1}{10+\frac{1}{5}}}$. These are actually the second and the third partial fractions from the progress of $3\sqrt{3}$ like a continued fraction. And because of Christiaan Huygens we know that $\frac{265}{51}<3\sqrt{3}<\frac{1351}{260}$. Archimedes also found the famous approximate value of $\pi - \frac{22}{7}$, which is the first partial fraction of π. Before that the ancient Egyptians used 3 for the ratio between the length of the circle and its diameter.

THE PROBLEM WITH THE CALENDAR

We know that the astronomical year doesn't have a whole number of days and nights. That causes problems in our everyday life but at the same time it generates a lot of very interesting math problems. By an astronomical method it is established that the duration of each astronomical year is 365 twenty-four hours, 5 hours, 48 minutes and 46 seconds. That equals $365\frac{20926}{86400} = y$ days. It is resembled like a continued fraction this way:

$$y = 365 + \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{64}}}}}}$$

In the Egyptian calendar and the Old Persian calendar each year included 365 days, which is K_0 of y. But some time passed after the creation of these calendars and the winter months, according to them, became summer months. That way people understood that the duration of the year was more than 365 days. In the old Julian calendar the duration of the year was $365\frac{1}{4}$ days. That equals K_1 of y. They made every fourth year 366 days long and the other ones remained 365. But again the winter months, according to the calendar, became summer months. This calendar was also not accurate enough. Then the Gregorian calendar was made and the duration of the year was considered $365\frac{8}{33}$, which is K_3 of y. This time the calendar was very accurate, its difference with y is just 19 seconds. Today we use the Gregorian calendar because of its accuracy.

REPRESENTATION OF π AS A CONTINUED FRACTION AND THE HISTORY OF FINDING ITS FIRST PARTIAL FRACTIONS

We owe the finding of the first partial fraction $\frac{22}{7}$ of π to Archimedes. Another popular approximation of π is $\frac{355}{113}$. This is the third partial fraction of π and it is known as the number of Metsius. The interesting thing is that it starts to differ from π from its seventh digit after the decimal point. The Metsius number was found during the fifth century from Tzu Chun Dzhu – a great mathematician and physicist, and his son. It was rediscovered in the Middle ages and mistakenly named after Metsius. π represented like a continued fraction looks like that:

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}$$

Because of the big number 292 in the continued fraction the fourth partial fraction of π is very accurate (it starts to differ from π from its tenth digit after the decimal point!).

THE INFINITE CONTINUED FRACTION WHICH CONTAINS ONLY THE DIGIT 1 (φ(THE GOLDEN SEQUENCE))

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}=\varphi$$

This continued fraction has a very interesting feature. Its k^{th} partial fraction is the ratio between the $k+2^{nd}$ Fibonacci number and the $k+1^{st}$ Fibonacci number. For example $K_0=1=\frac{F_2}{F_1}=\frac{1}{1}$; $K_1=2=\frac{F_3}{F_2}=\frac{2}{1}$; $K_3=$
$=\frac{3}{2}=\frac{F_4}{F_3}$; ... This continued fraction looks so simple but it is yet very significant and complicated. It looks that simple because φ is equal to $\frac{1+\sqrt{5}}{2}$ and that proves the fact that φ is irrational which means that the continued fraction above is endless.

FIBONACCI SEQUENCE AND FIBONACCI NUMBERS

The Fibonacci sequence is a sequence, such that each number is equal to the sum of the two previous ones in the sequence and its first elements F_0 and F_1 (the numbers that form the sequence are commonly denoted by F_n and are called Fibonacci numbers) are equal to 0 and 1, respectively.

THE MOST IRRATIONAL NUMBER?!

In school our teachers teach us that there are rational and irrational numbers and that there isn't a difference between the irrational numbers. Well actually there are some numbers that are more irrational than others like some infinities are bigger than others. We know that the partial fractions are the rational numbers that best approach the irrational number represented like a continued fraction. So the less accurate the partial fractions are, the more difficult it is to find an accurate fraction that approaches the number. So it is harder to find a fraction that has a pretty close meaning to the meaning of the continued fraction, the irrational number. And the smaller the elements of the continued fraction are, the less accurate the partial fractions are. Now, after some reasoning, we understand that the most irrational number is $1+\frac{1}{1+\frac{1}{1+\ddots}}$ which is actually equal to φ, so the most irrational number is φ. Lets compare the accuracy of the partial fractions of φ and π. We are going to see that the partial fractions, with the same index, of π and φhave a big difference in the accuracy. This is caused by the small elements, which are contained in the continued fraction of φ.

$\pi \approx 3.1415926535898$			
Partial fraction	*fraction*	*approximate value*	*accuracy*
K_0	*3*	*3*	*first digit*
K_1	$\frac{22}{7}$	*3.14285714…*	*2 digits after the decimal point*
K_2	$\frac{333}{106}$	*3.14150943…*	*4 digits after the decimal point*
K_3	$\frac{355}{113}$	*3.141592035…*	*6 digits after the decimal point*
K_4	$\frac{103993}{33102}$	*3.14159265301…*	*9 digits after the decimal point*

$\varphi \approx 1.61803398875$			
Partial fraction	*fraction*	*approximate value*	*accuracy*
K_0	*1*	*1*	*first digit*
K_1	*2*	*2*	*zero digits*
K_2	$\frac{3}{2}$	*1.5*	*first digit*
K_3	$5/3$	*1.666667*	*1 digit after the decimal point*
K_4	$\frac{8}{5}$	*1.6*	*1 digit after the decimal point*

We can see that between the fourth columns of the two tables there is a big difference between the numbers. That way we showed that our observations are reasonable and that the most irrational number is φ.

SOLVING QUADRATIC EQUATIONS USING CONTINUED FRACTIONS

Continued fractions are most conveniently applied to solve the general quadratic equation expressed in the form of a monic polynomial:

$x^2 + bx + c = 0$

which can always be obtained by dividing the original equation by its leading coefficient. Starting from this monic equation we see that:

$$x^2 + bx + c = 0$$

$$<=> \ x^2 + bx = -c$$

$$If \ \ x \neq 0 <=> x + b = \frac{-c}{x}$$

$$<=> x = -b - \frac{c}{x}$$

But we can continue applying the last equation to itself to obtain

$$x = -b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{-b - \ddots}}}}$$

If this infinite continued fraction converges at all, it must converge to one of the roots of the polynom x2 + bx + c = 0. Unfortunately, this particular continued fraction does not equal a finite number in every case. We can easily see that this is so by considering the quadratic formula and a monic polynomial with real coefficients. If the discriminant of such a polynomial is negative, then both roots of the quadratic equation aren't real. In particular, if b and c are real numbers and b2 - 4c < 0, all the convergents of this continued fraction "solution" will be real numbers.

And here is the general theorem:

By applying a result obtained by Euler in 1748 it can be shown that the continued fraction solution to the general monic quadratic equation with real coefficients $x^2 + bx + c = 0$, given by:

$$x = -b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{-b - \cfrac{c}{-b - \ddots}}}}$$

converges or not depending on both the coefficient b and the value of the discriminant, $b^2 - 4c$. If b = 0 the general continued fraction solution is totally divergent; the convergents alternate between 0 and ∞. If b ≠ 0 we distinguish three cases:

1. If the discriminant is negative, the fraction diverges by oscillation, which means that its convergents wander around in a regular or even chaotic fashion, never approaching a finite limit.
2. If the discriminant is zero the fraction converges to the single root of multiplicity two.
3. If the discriminant is positive the equation has two real roots, and the continued fraction converges to the larger (in absolute value) of these. The rate of convergence depends on the absolute value of the ratio between the two roots: the farther that ratio is from unity, the more quickly the continued fraction converges.

When the monic quadratic equation with real coefficients is of the form $x^2 = c$, the general solution described above is useless because division by zero is not well defined. As long as c is positive, though, it is always possible to transform the equation by subtracting a perfect square from both sides.In symbols, if $x^2 = c\,,(c > 0)$ just choose some positive real number p such that $p^2 < c$. Then we obtain:

$$x^2 - p^2 = c - p^2 \;<=>\; (x-p)(x+p) = c - p^2$$

$$x - p = \frac{c - p^2}{x + p} \;<=>\; x = p + \frac{c - p^2}{x + p}$$

But we can continue applying the last equation to itself to obtain:

$$x = p + \cfrac{c - p^2}{2p + \cfrac{c - p^2}{2p + \cfrac{c - p^2}{2p + \ddots}}}$$

and this transformed continued fraction must converge because all the partial numerators and partial denominators are positive real numbers.

In conclusion continued fractions have a wide application in mathematics and they are its engine. They are very useful and just beautiful.

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MP52. TYPES OF SPACES AND TRANSFORMATIONS BETWEEN THEM

Teodora Stanislav Vasileva
125th SU "Boyan Penev", Bulgaria

ABSTRACT

Below are presented the different methods of transformation between different spaces, both in terms of their dimension and the types of spaces. Briefly shown is the general theory of coordinate systems on a surface. Addressed are the properties of the main geometric spaces: Euclidean, elliptical and hyperbolic. It is shown that physical space is hyperbolic, and we live on an elliptic surface. Also shown is that human visual space is hyperbolic. Further the transformation of physical space to geometric and mapping space is considered, and finally, the practical application of ellipsoid geometry, in the development and use of cartographic projections, has been examined.

We intuitively assume that the space around us is Euclidean: homogeneous, continuous and isotropic. I will show that this is not true and that in our everyday life we often use non-Euclidean spaces.
I will look at the three spaces in their dimensionality and curvature

1. EXAMINING DIFFERENT SPACES DEPEDING ON THEIR DIMENSION

We usually work with two-dimensional and three-dimensional spaces, that is why they will not be presented. It would be more interesting if we are going to look at two one-sided surfaces.

1.1 MÖBIUS STRIP

Figure 1. Möbius strip [1]

Möbius strip (Möbius band, or Möbius loop) is the first one-sided surface to be found and investigated. It is a three-dimensional construction formed from a long rectangular strip which twists 180 degrees and then its two ends are glued together (Figure 1). This form has only one edge and one surface on which there are no left and right directions. A Möbius strip can be painted with a brush without lifting it off its surface.When we cut it along its centreline, instead of two Möbius strips, we will get a double-sided strip, but double twisted. If we cut a Möbius band at a distance of 1/3 of the edge, we will get two strips - a narrower Möbius band and another - a long double twisted strip (Figure 2).

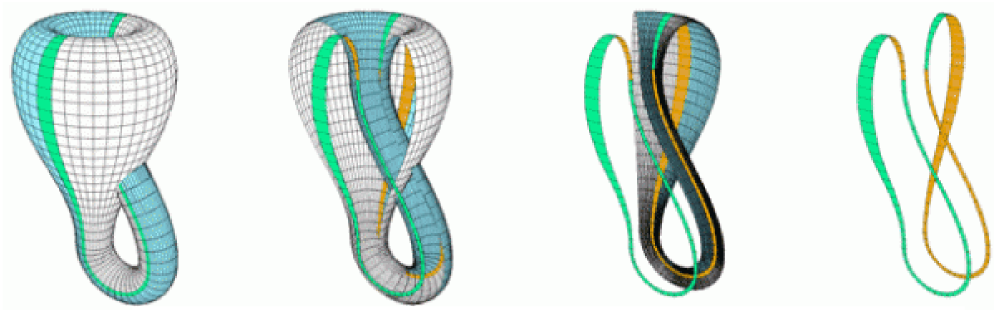


Figure 2. The two Möbius bands of a Klein bottle are connected by an ordinary two-sided band whose back and front sides are colored white and blue respectively [2]

August Ferdinand Möbius discovered the band at the same time as the German mathematician Johann Benedict Listing, but Möbius studied it more thoroughly.
Möbius band inspires artists. Moriz Escher particularly likes this loop and many of his lithographers are dedicated to this mathematical object. One of the most famous, Möbius Strip II, depicts ants crawling on the Möbius stript. The strip is an inspiration for the sign of infinity (∞).

1.2 KLEIN BOTTLE

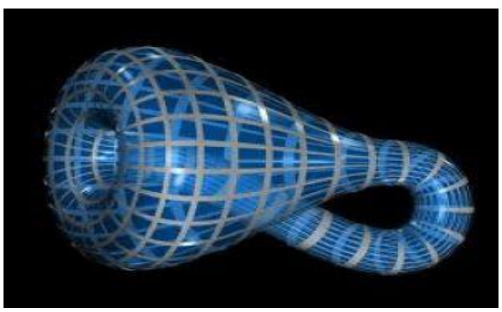


Figure 3. Model of the Klein bottle [3]

The Klein bottle (Figure 3) is a two-dimensional surface that has only one side, i.e., it can not distinguish between "internal" and "external". It can not be constructed in a lower than four-dimensional space, although an idea of it can be obtained from two-dimensional and three-dimensional images. The object was described for the first time by the German mathematician Felix Klein in 1882. At first Klein called it “surface” ("Fläche"), but it was translated incorrectly to English as “bottle” ("Flasche"). This mistake is easily explained by the familiar image of the surface, which looks like a bottle whose bottom with a hole is curved and passing through the wall of the bottle, where it merges again with its throat. The Klein bottle is an example of surface which is both unilateral and closed. It can be made from Möbius strip (Figure 2).
The Klein bottle can be successful constructed in four-dimensional space, in which the self-intersection and the inevitable opening in the surface, which impose the limits of the two-dimensional and the three-dimensional images, will not occur.

1.3. MULTI-DIMENSIONAL SPACES

Until the end of the 18th century, the idea of spaces with more than three dimensions was considered as "heretical". In the nineteenth century, interest in the subject grew (the New Zealand professor Duncan Somerville published in 1911 a bibliography with 1832 publications on n dimensions). The most remarkable results were achieved by Bernhard Riemann in his work "On the hypotheses which underlie geometry" from 1854.

In multidimensional Euclidean space, all axioms apply except : „If two planes have a common point they have a common line." If there are two planes that have only one common point, the space is at least four-dimensional. The term "plane" in multidimensional space is defined such that a set of points, which, along with each of its two points, also contains the line passing through them.
In the coordinate approach, to get the next dimension, we just add a new coordinate.

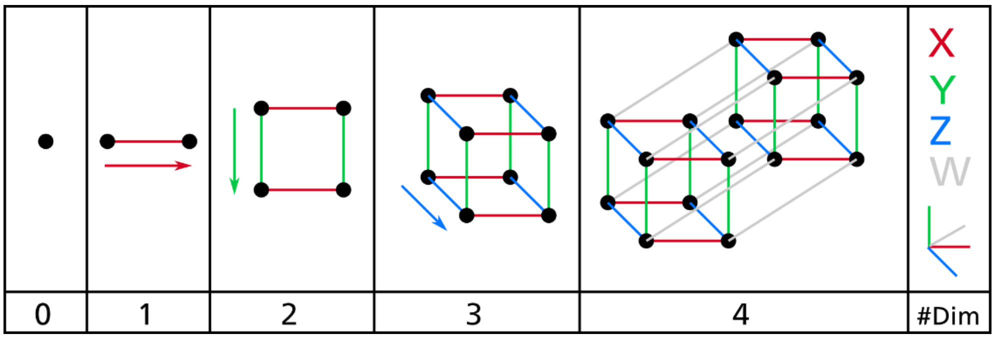


Figure 4. From 0D to 4D [4]

2.DIFFERENT SPACES, DEPEDING ON THE SPACES THEY ARE ON (DIFFERENT TYPES OF GEOMETRIES)

Depending on the curvature of space, the geometries are divided into two main types: Euclidean (curvature 0) and non-euclidean (with a curvature, different from 0).

2.1. EUCLIDEAN GEOMETRY

At first, geometry was a practical science that studied distances, areas and volumes, Euclidian geometry was developed in the III century BC. In Egypt, Before Euclid they used mathematically incorrect but practically satisfying geometry. Euclidian geometry is a mathematical system developed in Egypt by the ancient Greek mathematician Euclid of Alexandria. It is based on 22 axioms and 5 postulates. Euclid's Elements is the earliest completed text about geometry. Euclidean geometry is modeled following the concepts of point, line, plane.
Euclid's "Elements" begin with planar geometry (planimetry) and contains the first examples of a mathematical proof. Elements include spatial geometry in three-dimensional space, also called stereometry.
For more than 2,000 years, the geometry of Euclid has not expanded, because no one imagined the existence of other types of geometry.
The geometries after Euclid were hampered by the complexity of the Fifth Postulate, and for centuries mathematicians were trying to prove it as a theorem based on the other four postulates. The original formulation of this postulate is:

"If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles less than two right angles occur." (Figure 5).

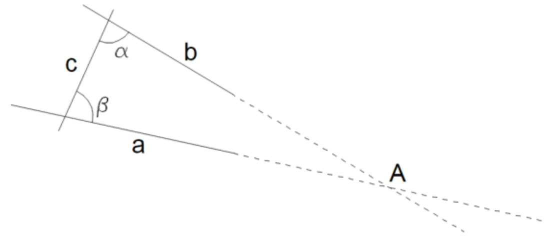


Figure 5. If the sum of the interior angles α and β is less than 180°, the two straight lines, produced indefinitely, meet on that side

Present-day mathematicians give simpler equivalent formulations of this "parallel postulate":
"In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point (Parallel postulate or parallel axiom)."
This is the last version of the postulate (made in the 18th century by John Playfair). The first formulation of this axiom was for non-intersecting straight, instead of parallel.
By the 20th century, no one had renounced the axiom and because of that no other geometries were found until then.

2.2. NON-EUCLIDEAN GEOMETRY

Non-Euclidean geometry is a common term for geometries that are different from the Euclidean geometry. The key difference between them is in the nature of parallel lines. They are divided into two subsets: **hyperbolic**, with negative surface curvature, and **elliptical**, with a positive surface curvature.

2.2.1. HYPERBOLIC GEOMETRY

In 1829, Nikolai Lobachevsky published a monograph on Hyperbolic Geometry. In Russia, they did not understand the idea of non-Euclidean geometry. The Russian Academy of Sciences gave it a negative rating, and a contemporary magazine wrote that it lacked elemental common sense. At about the same time, the Hungarian mathematician János Bolyai also wrote a treatise about Hyperbolic Geometry, which was published in 1832 as an attachment to his father's work. The great mathematician Carl Friedrich Gauss (1777-1855) read it and revealed to Bolyai that he had received the same results before him but did not publish them for fear of misunderstanding.

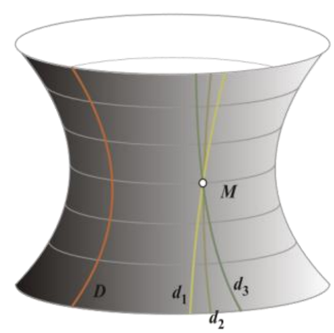


Figure 6. On this hyperboloid it is shown that d_1, d_2 and d_3 do not intersect line D [5]

In the formulation of their Fifth postulate: "…at least two lines parallel that do not intersect the given…" the term "parallel line" is not used. In this way, the axiom is wrong because, as shown in Figure 6, more than 1 line passes through the point. In today's formulation of the axiom (which is shown in 2.1), it is again wrong because these lines are divergent.

2.2.2. ELLIPTIC GEOMETRY

The name "elliptical" is misleading. It does not mean any direct connection to the curve, called the ellipse, but just an analogy. The simplest model of elliptic geometry is a sphere where the lines are large circles
(like the parallels of the globe).
This geometry has a special case in which the parallel axiom is true: the rotational ellipsoids. They are obtained by rotating the ellipse around one of its axes.

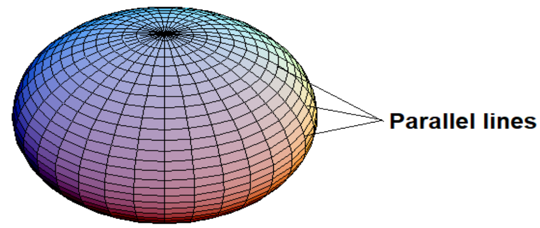


Figure 7. Latitude and longitude lines on a rotational ellipsoid

2.3. MAIN DIFFERENCES

The main difference between Euclidean and non-Euclidean geometric patterns is in understanding the very essence of space. Because of the different curvature, in non-Euclidean geometries, space is not homogeneous (points in space are not identical), whereas in Eucledian geometry it is.
Also figures are different in each geometry. The triangle, which we usually draw with three straight lines (curvature 0), "expands out" to a triangle with three arcs (positive curvature) in elliptical geometry and “shrinks inwards” to a triangle with three arcs (negative curvature) in hyperbolic geometry (Figure 8) While Euclidean geometry includes some of the oldest known mathematical views, non-embedded geometries were not widely recognized until the 19th century. At that time, the belief that the universe was working on the principles of Euclidean geometry was widely acceptted.

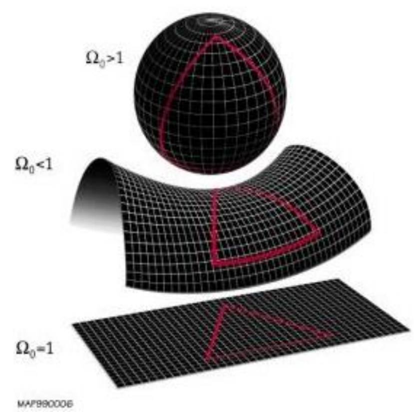
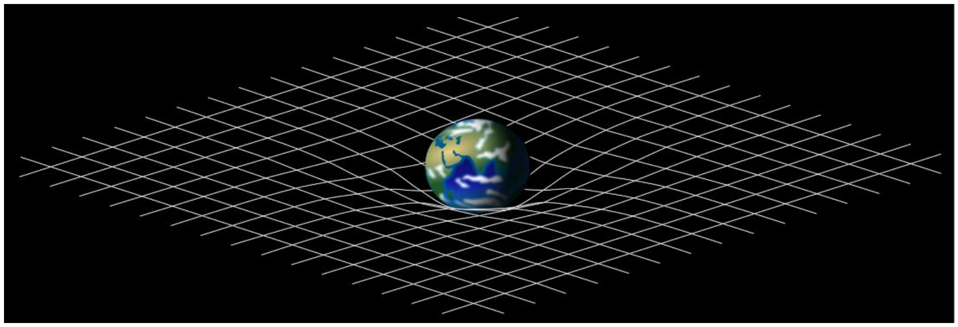


Figure 8. The triangles in the different geometries. [7]

3. THE SPACE AROUND US

According to the Special Theory of Relativity, we live in a hyperbolic world (Figure 9). Stars and planets have an elliptical surface. We do not see the curvature because of the very small scope of our eyes. Our eyes give us two different flat images. They are curved because the inner surface of the eye is elliptical (Figure 10).
Our brains corrects the curvation of the images obtained from the eyes and makes from them a three-dimensional model of the world around us. This model is not Euclidean. Research has shown that peripheral vision shortens distances. When watching terrain from a height, our brain perceives distances to be shorter then they realy are (Figure 11).

Figure 9. Two-dimensional image of space-time distortion. [8]



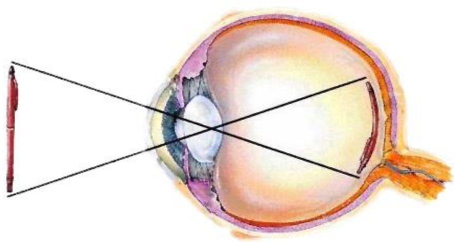


Figure 10. How does the picture reflect in the eye.[11]

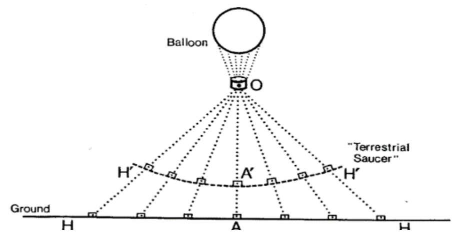


Figure 11. How do we see the Earth from a balloon [11].

4. GENERAL THEORY OF A COORDINATE SYSTEM ON CURVED SURFACE

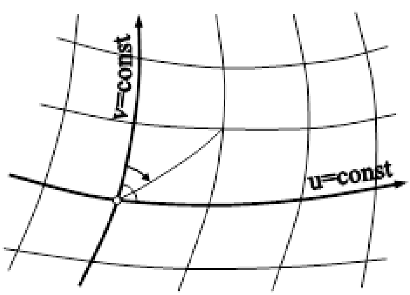


Figure 12. General coordinate system on a curve surface

On each curved but smooth surface, two systems of continuously numbered lines can be constructed such that each point (P) of the surface is contained within one and only one line from each system. The combination of the two systems allows to uniquely describe the location of each point on the surface by the numbered lines. Therefore, the aggregation of two systems of continuously numbered lines on a smooth surface in which each line on one surface crosses each line from the other surface at exactly one point is called a general coordinate system on a surface (Figure 10).

Let us denote the lines of one system with **v**, and the lines of the other system with **u**. The numbers of the lines passing through each point represent a ordered pair **(u, v)**, representing the curvilinear coordinates of the point. When moving along a v coordinate line, **v = const**, only the coordinates in the system **u** change. That is why this coordinate line is called **u** line. Similarly, the coordinate line **u = const** is called **v** line, because along its length **u** is not changed, and at each point the coordinate **v** is different. As the positive direction of the line **v = const**, the direction of the coordinate increase **u** is assumed, and the positive direction of the line **u = const** is the direction of the coordinate increment **v**. The angle between the positive directions of the coordinate lines is noted with **ω**. The concept of the **indicated angle (T)** of the oriented direction in point **P** means the angle that the oriented direction makes with the positive direction of the coordinate line **v = const**, passing through **P**.

If a rectangular coordinate system is introduced, then **P** will have rectangular coordinates **x**P, **y**P, **z**P. The connections between the curvilinear and the rectangular coordinate systems can be expressed by the functions:
x = fX(u, v);
y = fY(u, v);
z = fZ(u, v).

4.1. CARTOGRAPHIC PROJECTIONS

The transformation of terrestrial space into a cartographic iprojection is made in several steps:
First, points of the Earth's surface are translated to the geoid (a surface that approximately concurs with the global sea level and is below the continent level). This eliminates altitude (but we should remember it), which makes the following transformations easier.
Due to its complexity, the geoid is not mathematically defined. Therefore, the second step is to transfer the points to a rotational ellipsoid. For each part of the Earth's surface, a different ellipsoid may be chosen that best fits the given part.
In the third step, the elliptic a space is transformed to Euclidean space by mathematical functions:
x = f1(φ,λ) and **y = f2(φ,λ)**, (where φ and λ are respectively latitude and longitude)
by which the correspondence between a point of the ellipsoid with geographic coordinates is expressed **(φ, λ)** and its reflection on the map with rectangular coordinates (x, y).

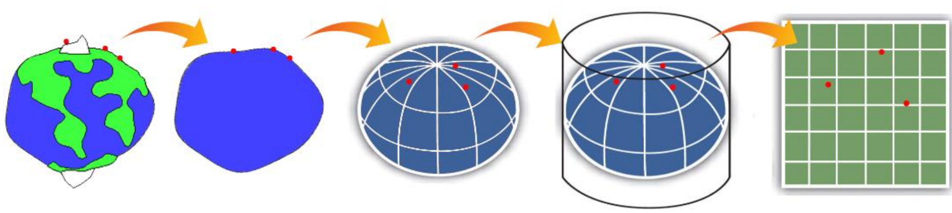
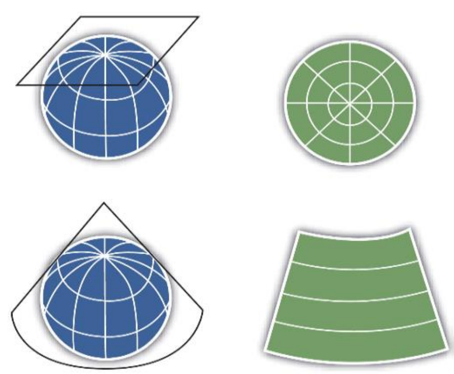


Figure 13. Transformation from the Earth surface to map.



The solution to the problem with the transformation is not unambiguous, so different functions f1 and f2 can be defined so as to obtain a projection with desirable qualities.The difference is in the deformations obtained in the process of projecting of angles, lengths and areas. Sailors who are compass-oriented should use maps in which angles are depicted without distortion; land administration would like the areas depicted on the map to be true, and car travelers would prefer that the exact distance are shown on the map.
Geometrically, the projection of ellipsoid points in Euclidean space is limited to transferring ellipsoid points to the plane by using mathematical figures that can unfold in the plane without deformations. For this purpose a plane, cone and cylinder are used. Therefore, the projections are divided into azimutal, conical and cylindrical respectively (Figure 11).

The last part of transforming the real earth surface onto the plane of the map is scaling the image so that when making measurements on the map, real distances and areas of the area can be determined.

We are living in a non-Euclidian world, as we saw above. Every day we are working with non-Euclidian geometry more than we realise. But Mathematics is still working mostly with Euclidian geometry. There are a few people working on the transformations between Elliptic and Hyperbolic geometries and Hyperbolic and Euclidian geometries. But we can either wait for other people to find out solutions to these problems or discover them ourselves.

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MP54. MATHEMATICS IN CRYPTOGRAPHY

Khalid Mashal, Talemwa Nanyange Kyambadde
Turku International School, Finland

ABSTRACT

Cryptography is the writing of codes to protect information through the use of encryption and decryption. Cryptography is a set of mathematical functions to create algorithms called ciphers that turn basic text (often referred to as plain text) into 'ciphertext'. Ciphertext is hard to decipher as the original letters are replaced with letters, numbers and symbols through mathematical equations, thus making the information encrypted and transformed into formats that cannot be recognized by unauthorized users. Most encryption is based on number theory, exponential maths, Euler's theorem and a lot of abstract algebra that make the ciphers. With this method, information such as passwords, emails, private data, and even cryptocurrency such as bitcoin can be protected. Cryptography dates back centuries; one of the earliest being the Caesar cipher which is one of the simplest ciphers for encrypting. These ciphers were handwritten messages coded so unintended recipients would not be able to understand the message. Even though these ciphers are very simple and therefore not very reliable, they can still be used to introduce cryptography to students. Today, most cryptography is computerized and therefore more intricate as that used by organizations, such as the military, banks and governments to protect sensitive information.

INTRODUCTION

Cryptography is the art of writing and solving codes; the practice and process of converting ordinary information, data or text into an unintelligible form and vice versa (Encryption and decryption), using ciphers.
This paper will show how mathematics applies to cryptography, as mathematics is the base for most forms of cryptography. Ciphers, which make cryptograms, are made of mathematical functions and algorithms, so without maths, cryptography would not be attainable.

THE HISTORY OF CRYPTOGRAPHY

Cryptography dates back thousands of years. The first forms of cryptography were handwritten secret messages, so parties could communicate discreetly. It was used to convert normal messages into hand-written secret code that would only be understood by the sender and the recipient. These were used by many civilizations, the Egyptians and the Greek being some of the earliest traced. Through the years, cryptography developed, to writing secret coded messages with machines.
Nowadays, cryptography is mostly digital, being used to encrypt information and communication to prevent third parties for example, the public and other organizations from unintended access. Every day, people use technology, send text messages and emails, save data on the cloud, and

more. All this information is protected by an encryption barrier, keeping the data safe from hackers that would use the information for malicious purposes or to harm a person, society or stability of an organization. Examples where encryption is applied include copyrighting, digital rights and authentication.
But the encryption may sometimes be weak and assailable, making it easy for adversaries to infiltrate and break the encryption barrier, and find their way into information unintended for them to have access to.
This all depends on the strength of maths used to make the encryption. So, a solution to this problem would be to increase the complexity of the cipher by using more advanced mathematics.
The process of encryption can vary in difficulty levels. One of the simplest ciphers that exists is the Caesar Cipher. Named after Julius Caesar himself, he used it to communicate in private with lieutenants and allies without his messages being interfered with by others.

EXAMPLES OF MATHEMATICAL CIPHERS

The Caesar cipher uses a simple shift key. Here is an example.
Let ***n*** be the variable.
The cipher uses a shift key, where the alphabet is shifted by ***n***, and each letter is replaced with a letter that is ***n*** steps away from. Each letter then has a new meaning.

Normal alphabet	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Shifted alphabet	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s

Figure 1 - Cipher Alphabet

As shown on the diagram, a letter is shifted by ***n***, which in this case is 19, to the left (reverse shift). It is important for the receiver to know the key (value of ***n*** and in which direction it is set to), to be able to decipher the encrypted text. Then using the new alphabet, one is able to write hidden messages.

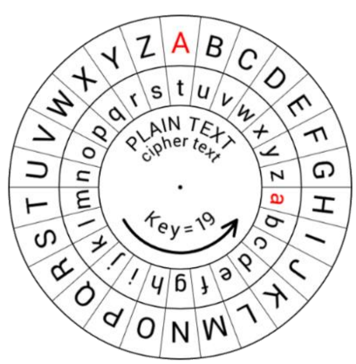

Figure 2 - Cipher Wheel Diagram

Plain-Text →	I M P E N E T R A B L E
Cipher-Text (applying the same shift as shown in the diagram above) →	b f i x g x m k t u e y

Original text is referred to as *plain-text* and the coded message as *cipher-text.*

Nowadays, codes such as the Caesar Cipher are not used, seeing that its design does not meet the requirements of the modern world, since computation power has increased ever since. So, more complicated ciphers have been formed to make stronger, more impenetrable encryptions. And even today, more complex codes and ciphers are being invented. These codes are far too difficult for humans to manually write or crack, so machines and computers are used to create them.

The foundations for cryptography are computer science and mathematics. Ciphers are simply mathematical algorithms. For example, using the Affine cipher which works like a linear function, where the input is multiplied by a number and then added by another number.

The secret message

E X A M P L E

Each letter can be given a number, like its place in the alphabet.

E	X	A	M	P	L	E
4	23	0	12	15	11	4

We can then multiply each number. In this case, by 3.

	E	X	A	M	P	L	E
	4	23	0	12	15	11	4
X 3	12	69	0	36	45	33	12

Then we can add a number to each of them. Here, adding 11.

	E	X	A	M	P	L	E
	4	23	0	12	15	11	4
X 3	12	69	0	36	45	33	12
+ 11	23	80	11	47	56	44	23

We can then calculate all the answers by mod 26. The mod calculates the remainder when one number is divided by another. Here, we divided specifically by 26 because there are 26 letters in the English alphabet. if we use any other language with a different amount of letters, then that number would be the modulus.

	E	X	A	M	P	L	E
	4	23	0	12	15	11	4
X 3	12	69	0	36	45	33	12
+ 11	23	80	11	47	56	44	23
mod 26	23	2	11	21	4	18	23

The numbers can then be converted back to the alphabet, therefore making our cipher-text.

	E	X	A	M	P	L	E
	4	23	0	12	15	11	4
X 3	12	69	0	36	45	33	12
+ 11	23	80	11	47	56	44	23
mod 26	23	2	11	21	4	18	23
	X	**C**	**L**	**V**	**E**	**S**	**X**

And that becomes our coded message.

X C L V E S X

The cipher used here would be

$$Ex = (ax + b) \bmod m$$

E = Encryption
a = 3
b = 11
m = 26
x = the number representing a letter

As seen above, the cipher is a mathematical equation.
To decrypt the coded message, we must perform the inverse function.

$$D(x) = c\,(x - b) \bmod m$$

Here, *c* is the modular multiplicative inverse of *a*, which equal to 9.

	X	**C**	**L**	**V**	**E**	**S**	**X**
x	23	2	11	21	4	18	23
$9(x-11)$	108	-81	0	90	-63	63	108
$9(x-11)$ ***mod 26***	4	23	0	12	15	11	4
	E	**X**	**A**	**M**	**P**	**L**	**E**

As shown above, this leads us back to our original text, ‘EXAMPLE’.
These are just two examples of simple ciphers that are based on mathematics and can easily be manually decrypted. There are various, more complex ciphers that are generated by computers. Again, all these are based on maths.

CONCLUSION

Cryptography has evolved over time from simple mathematical functions, such as the Affine cipher, into more complex ones generated by computers. Mathematics and computer science now form the foundation of modern cryptography. It is therefore imperative to say that these two subjects should be strengthened in school.

RECOMENDATION

In this modern world, our lives now and in the future greatly depend on technology and information. So, it is essential for each one of us to have basic knowledge of how cryptography functions. It is therefore important to include cryptography in schools starting from an early stage so that everyone is able to understand how to keep our data safe. The mathematics behind the simplest encryption schemes suits even in elementary school as it is only shifting the alphabet. For more advanced encryption schemes, one needs a more sophisticated mathematical basis which is obtained at higher levels of education.

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MP55. MATHEMATICS AND PHILOSOPHY

Juan A. Monge-Navarro Oterp
Turku International School, Finland

ABSTRACT

The following work will attempt to present to the reader the topic of Quasi-Empiricism in Mathematics. Normally mathematics and Empiricism wouldn't go together that well because in its most basic form Empiricism is a philosophical theory that basis all knowledge through sensitive experience. But Quasi-Empiricism attempts to underline the relationship between math with physics, social science and computational science. Also through this theory we could approach problems in Empiricism such as realism and necessity of application. For philosophers like Laszlo Kalmar all mathematical axioms are in a way extracted from empirical facts. This would mean that empiricism is essential to all mathematical axioms and since axioms are the building blocks of mathematics Empiricism is something which we would have to understand if we want to fully understand Mathematics. So in a nutshell the following work will present to the reader the influence of Empirical thought in Mathematics.

PHILOSOPHY

For many people philosophy is an unapproachable topic due to its complexity, but philosophy is not complicated once we understand the 4 major branches that compose it. You might be wondering why explain philosophy, well since in this paper we will present the influence and application of empirical thought in mathematics we should have clear what is philosophy.
The word "Philosophy", in Greek, "φιλοσοφία" (_philosofia_), literally means "the love of wisdom". The meaning of the word gives us a sense of what philosophy is, a search of knowledge and also the questioning of that knowledge. As it was mentioned before philosophy has 4 major branches: Logic, Ethics, Metaphysics and Epistemology.
Logic, The word Logic derives from the Greek "_Logos_"[1] which mean "reason" we can assume by the origin of the word that it has something to with reason and yes Logic is the branch which studies reasoning using logic as the main tool. It also analyses if an argument is valid on what's trying to convey and Logic also allows us to identify good reasoning from bad reasoning. Philosophers which have excelled and transcended Logic have been for example: Aristotle and Hillary Putnam.[2]

Ethics, Ethics is a branch of philosophy which in its most basic form is the study of if an action is right or wrong. Ethics is usually divided into 3 major topics in Ethics: Meta-ethics, Normative

[1] _"Logic (n.)." Index, www.etymonline.com/word/logic._
[2] _"LOGIC | Meaning in the Cambridge English Dictionary." Cambridge Dictionary, dictionary.cambridge.org/dictionary/english/logic._

ethics and Applied ethics. Meta-ethics studies where our moral laws and ethical standards come from as well as their meaning. Normative ethics is in charge of getting to moral standards which define what's right and wrong. And finally Applied ethics is ethical theories applied in the real world in problems such as Abortion.3 Some philosophers that have formed excellent ethical theories are: Immanuel Kant, John Stuart Mill, Friedrich Nietzsche and Aristotle.

Metaphysics, The word "metaphysics" in itself might get us thinking that it has to do with physics somehow but if we take a closer look at the prefix that the word contains, "meta-", we can see that the literal meaning of the word metaphysics is beyond or over physics (because the prefix "meta" means over in Greek). So in short the purpose of metaphysics is to reach to what's beyond physics and try to reach for things like the meaning of being4. Some philosophers that have studied Metaphysics and formed excellent works within the boundaries of it have been philosophers such as: Plato, René Descartes, Jean-Paul Sartre and Arthur Schopenhauer.

Epistemology, The word epistemology is a word that is composed by 2 words with a Greek origin and said two words translate to "episteme" knowledge and "logos" reason. From the meaning of said word we can conclude that it has something to do with knowledge and yes in fact Epistemology not only studies the nature and origin of knowledge but also its limits5. Most philosophers have laid their thoughts on knowledge and its origins but some philosophers that have laid remarkable ideas have been for example: John Locke, David Hume, Baruch Spinoza and Immanuel Kant

Empiricism

In epistemology, Empiricism is the doctrine within epistemology which theories´ that (in the most extreme form) all knowledge or what makes up that knowledge comes from experience. Empiricists believe or believed that our mind starts as a "blank tablet" and we imprint information on that "blank tablet" through sense data, information that is provided by the senses. In epistemology we use terms such as "A Priori" and "A Posteriori" to describe the nature of knowledge. In the case of Empiricism the knowledge is said to be acquired "A Posteriori" which means "from the former" (in Latin) and not "A Priori" means "from the earlier" (again in Latin). The official founder of empiricism is considered by many to be John Locke although many philosophers have contributed to this epistemological theory or rationalism at the time in which both where at conflict with the other one. Other Empirical philosophers have been for example: David Hume, George Berkley, and Arthur Wittgenstein.[6]

[3] *Internet Encyclopedia of Philosophy*, Internet Encyclopedia of Philosophy, www.iep.utm.edu/ethics/.

[4] *Welcome to Principia Cybernetica Web*, pespmc1.vub.ac.be/MEANMET.html.

[5] *Duignan, Brian, et al. "Empiricism." Encyclopædia Britannica, Encyclopædia Britannica, Inc., 22 July 2016, www.britannica.com/topic/empiricism.*

[6] *Honderich .T (1995) The Oxford companion to philosophy, New York, Oxford University Press (p.226-229)*

Since it is very much knowable that not all mathematical knowledge is or can be acquired empirically we need to use what is called a Quasi-Empiricism. A Quasi-Empiricism means that it's close to empirical but not quite it and that's why the prefix "Quasi-" (that means "close to")[7] is used. The feeling that mathematical knowledge is, at least in its most basic form, acquired empirically is shared by philosophers such as Hillary Putnam and John Stuart Mill and in this paper we will analyze different parts of mathematics and find evidence of the unmistakable sign that empiricism and mathematics are connected.[8]

Arithmetic

So that we can try to use Empiricism to analyze arithmetic knowledge we must have a ground set definition for what exactly is that area of mathematics that we call "Arithmetic". Arithmetics is a branch of mathematics which is concerned with numbers and yes in this definition I will include algebra and number theory as part of arithmetics for the sake of simplicity. For arithmetic I will explain the basic operation and numbers then for algebra I will explain equations and fractions and I will put 2 examples of the Empirical method used to "Quasi-prove" some conjectures in number theory.
Numbers are something that we have to have clear to begin with. Numbers as we know them are just symbols that are used to represent a quantity, except for numbers such as negative numbers and quaternions. The symbols are just symbols that represent ideas that cannot be tarnished such that the symbol of three: 3 cannot symbolize that I have 4 horses because we need the symbol that represents 4 to do it.
The most basic operations in mathematics such as: addition, multiplication, subtraction and division are all which we can acquire empirically. First addition, arithmetic facts such 2+2=4 looks very abstract but Aristotle would say that thesis such as the one before are just generalizations of the fact that I have 2 books here and 2 books there and I make the assumption that I have 4 books all together[9]. One could argue that very big numbers such as 34542 are numbers that a normal person won`t encounter in a real life scenario well with numbers such as these we can just imagine adding one by one till be get 34542.

$$\sum_{1}^{34542} 1 = 34542$$

And doing this is not empirical but its is an elaboration on an empirical acquired fact and this type of knowledge is very common in mathematics, such that we have empirical facts which we

[7] "QUASI- | Meaning in the Cambridge English Dictionary." *Cambridge Dictionary*, dictionary.cambridge.org/dictionary/english/quasi.

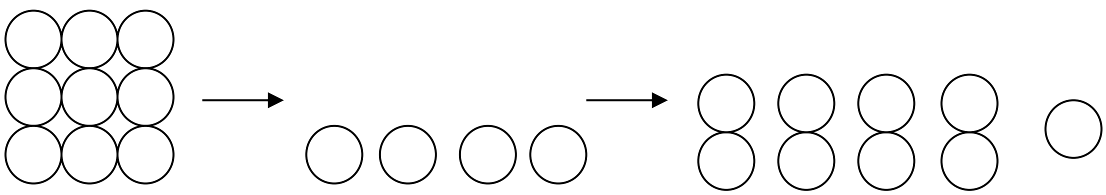
[8] "Kids.Net.Au - Encyclopedia > Quasi-Empiricism in Mathematics." *Kids.Net.Au - Encyclopedia >* encyclopedia.kids.net.au/page/qu/Quasi-empiricism_in_mathematics.

[9] *Honderich .T (1995) The Oxford companion to philosophy, New York, Oxford University Press (p.532-533)*

later build on with facts that have been acquired by using pure reason. Since I have shown how addition is an empirical fact ill move to subtraction, and subtraction is just a generalization of taking away things for example I have 4 horses and I take away 2, I am left with only 2 horses.
Then we can move on to multiplication, multiplications can just be seen as the generalization that multiplying can be seen as adding sets of stuff such as, if I am told to solve 3 x 4 then they are just telling me that I should add 4, 3 times or vice versa given that multiplication is commutative.

$$Y \times X = \sum_{1}^{Y} X$$

Division, now division is a bit more complicate but mostly because we have to use some pure reason to solve this kind of problems. Now imagine that in a group of 4 friends I have to give away 21 marbles. Now we can divide the 9 marbles by doing the next we go one per one and give them 1 marble each time and then for the result we will end up with a remainder and the solution to the division that we can know by seeing how many marbles does one kid have (figure 1).



(Figure 1)

The result we get is that we get the result of 2 and as a remainder 1. And if we use the operation of module we get the result of 1 (we get the remainder), such that: $9mod4 = 1$. This operation of module is also a side product of this division.

$\forall \{ Y \div X = Z, |Z| \in \mathbb{N} \mid Ymod\ X = 0\}$

$\forall \left\{ Y \div X = Z + \frac{K}{X} \rightarrow\ Ymod\ X = K, |Z| \in \mathbb{N} \right\}$

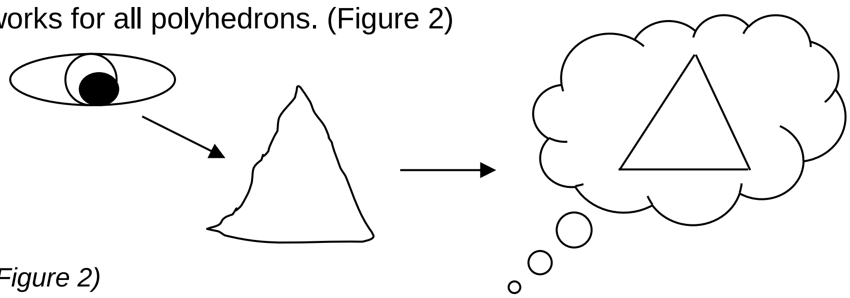
Now we will move on to algebra, algebra is just arithmetic's that use letters to represent unknown numbers but in algebra that value that cannot be tarnished such that in the equation 4+a=5, a will have a value of 1 because if not the equation enters in a contradiction. In the early ages what could be called "algebraic" problems were in fact solved geometrically and this way of solving problems by first seeing them or visualizing them can be called Quasi-empirical because we are using first empirical data to later solve it using reason.[10]
In algebra we are often confronted with negative numbers and yes in nature we aren't able to find things such in negative quantities but then again negative numbers are just elaborations on

[10] "Algebra." _Wikipedia_, Wikimedia Foundation, 19 Feb. 2019, en.wikipedia.org/wiki/Algebra.

the empirical acquired fact that I can have 3 apples but then in our mind we can elaborate on that fact to reach to a number such as -3.
Another important thing on number are factions like ½ or ¾ or all rational numbers. Now if we put the numbers as word we get phrases such as, one half or three fourths. If we take a closer look we can deduct that the declaration of one half implies the existence of a whole thus if we generalize it to the fact that all rational numbers are in fact just elaborations on the (again) the empirically acquired fact of 1.
And in Number theory we can find many Quasi-Empirical "Quasi-proofs" for conjectures. I say Quasi- because we still for some haven't proved them yet but the thing is that for example the Collaz conjecture and the Goldbach conjecture that have proven to works for trillions of numbers and this proves that we are doing using computers aren't trying to prove that the conjecture works for all numbers the computers are just simply looking for a number that doesn't work. And for example for Fermat's last theorem before it was proven by Andrew Wiles,[11] it was proven to be correct for a massive amount of numbers and these Empirical "quasi-proves" are an excellent example of the scientific method applied in mathematics. But there have been examples of theorems that have been proven by computers such as "The four color theorem"

Geometry

First so that we can fully understand geometry, we will define it. Geometry can be defined as the branch of mathematics which studies the relationship and properties of: lines, points, surfaces, planes and areas.12 The problem that one encounter in geometry when one is analyzing geometry to search for evidence for empirically acquired knowledge one finds that because many things in theory such as line, points or polyhedrons are things that are just not possible in the real world. But to solve this problem first we have to look at a bigger picture.
The bigger picture here would be for example look at what seems to look like a triangle. Well in reality if we zoom in we will find out that this so called "triangle" actually has millions of edges. So to get the picture of an actual triangle I our minds we have to first look at that "triangle" that we've mentioned before, then in our mind we idealize it into what we in geometry define as a triangle. And the beauty in it is that a triangle can only live in our imagination and this process works for all polyhedrons. (Figure 2)



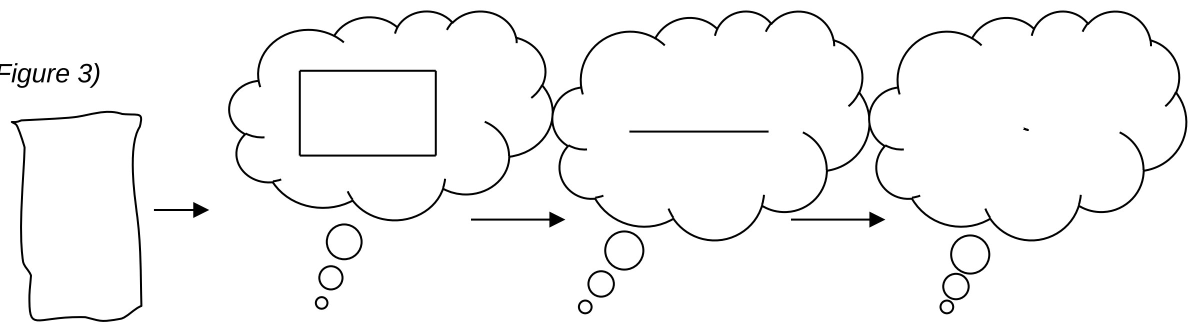
(Figure 2)

[11] "Hilary Putnam." *Wikipedia*, Wikimedia Foundation, 21 Nov. 2018, es.wikipedia.org/wiki/Hilary_Putnam#Filosof%C3%ADa_de_las_matem%C3%A1ticas.

[12] "Geometry." *Merriam-Webster*, Merriam-Webster, www.merriam-webster.com/dictionary/geometry.

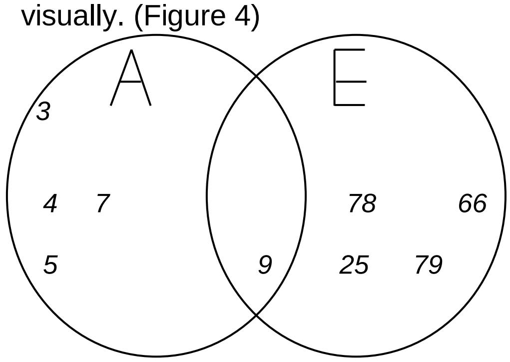
Now more abstract things such as lines or points require more thought. For line you first need to see what looks like a rectangle and then idealize it. Then that rectangle you take its width and take it to extreme such that your left with one single line of no width. And I know this is very abstract but that is why a line and a point are unimaginable and the best we can do is to just express those ideas as words or just use a line with very little width to express a line. Points can be explained as taking the line that we just made and just take the length to 0 because a point has length 0 and width 0. So a point can just be imagined the absence of everything with something. (Figure 3)

(Figure 3)



Set theory

Set theory is that branch of mathematics which studies sets. Sets are just collections of thing they can be mathematical objects or physical objects[13]. Set theory in practice is very empirical for finite sets because operations such as unions or intersections can be expressed and learnt visually. (Figure 4)



(Figure 4)

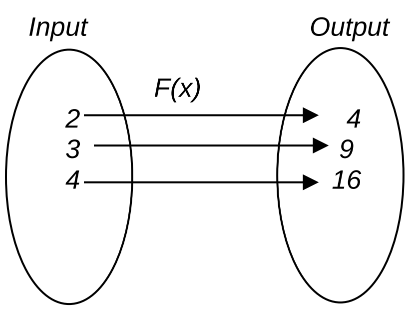
In the figure above we can see for example the sets A and E. The two sets have elements that compose said set it can be expressed such as: $A \in \{3,\ 4,\ 7,\ 5,\ 9\}\ and\ E \in \{9,\ 25,\ 78,\ 79,\ 66\}$. Also these two sets have an area in which they intersect. This area is called intersection, and

[13] Stoll, Robert R., and Herbert Enderton. “Set Theory.” *Encyclopædia Britannica*, Encyclopædia Britannica, Inc., 26 Feb. 2019, www.britannica.com/science/set-theory.

it's an operator in set theory which basically searches for what terms do a certain amount sets have in common, and its expressed such as: $A \cap E = \{9\}$. Then there is more operators like: union, set difference and symmetric difference.[14]
Unions, are just all the elements of the two sets combined. Ex. $A \cup E = \{3,4,7,5,9,25,78,79,66\}$
Set differences, it's an operator that takes away all the elements on a set that the other set has as well, basically the elements in one set but not in the other one. Ex. $A \setminus E = \{3,4,7,5\}$
Symmetric difference, it's an operator that expresses all the elements that are not in common. Ex. $A\Delta E = \{3,4,7,5,25,78,79,66\}$
An essential part of set theory is to learn it and operate it through with picture and diagrams due to its complexity and abstraction. An important thin to notice is that sets as collections of thing can be empirical facts because the set are visual objects more than abstract ones.

Calculus and functions

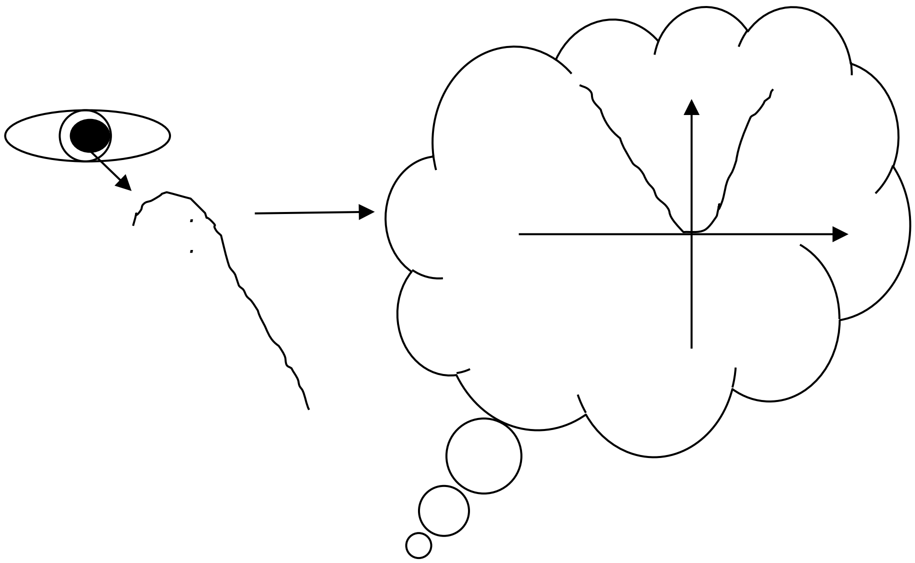
The last branch of mathematics that I will analyze will be Calculus and functions. Calculus is the area of mathematics which studies 2 branches within it: Differential calculus and Integral calculus. But first to study calculus we need to know what a function is and the nature of a function.
Functions, a function is basically something that takes an input value and spits an output value (figure 5). Descartes as he introduced the Cartesian coordinates he also introduced the mapping of functions. If we treat functions as curves or lines in a graph we can have the knowledge beforehand such that for example a parabola can just be seen as an idealization and generalization of for example the path that the piss does when we men piss because the path is a parabola (figure 6). Another example is the centenary that a rope forms when it's held with 2 poles.



_Ex._ $f(x) = x^2$

(Figure 5)

[14] "Set Theory." _Wikipedia_, Wikimedia Foundation, 18 Feb. 2019, en.wikipedia.org/wiki/Set_theory.



(Figure 6)

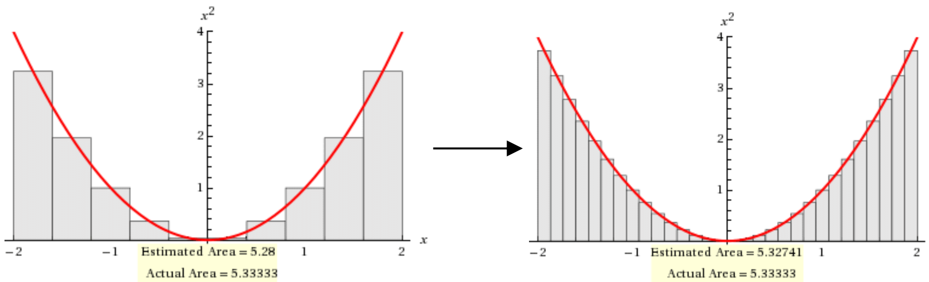
Not very good parabola

2 Other words which are essential to calculus are Infinity and Infinitesimal. Infinity can be defined as endless, never-ending, etc. Cantor extensively studied infinites but I will explain a more general infinite. An infinite cannot be grasped by our mind but the idea of a never-ending supply of things is something that somewhat can be grasped by the mind but I like to define infinite as the point when we have so many of something that is physically impossible to count them, this is a definition that could be expressed by the idea of the limit such that we go for a value that approaches infinite because infinity is just an impossible idea, e.g. $\lim_{n\to\infty} n$. So basically infinity is just an endless elaboration on the empirically acquired fact of 1 for example.

$$\sum_{1}^{\lim_{n\to\infty} n} 1 = \infty$$

And them the infinitesimal is just an approach towards the zero without actually reaching it and this in fact, very useful in integral calculus and also differential calculus. So it can be expressed as the idea of approaching certain value towards to zero without actually reaching it, e.g. $\lim_{n\to 0^+} n$.

Now we can move to integral calculus and differential calculus. Integral calculus is finding the area under a curve. If we want to give this a remotely empirical approach we must turn to the way Riemann himself tries to calculate the area under a curve by using his method of approximating definite integrals of Riemann sums:



[15]

"Let a closed $[a, b]$ be partitioned by points$a < x_1 < x_2 < \cdots < x_n < b$, where the lengths of the resulting intervals between the points are denoted, Δx_1 , Δx_2,..., Δx_n. Let x_k^* be an arbitrary point in the k_{th}. Then the quantity

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k \approx \int_a^b f(x)\, dx$$

is called a Riemann sum for a given function $f(x)$and partition, and the value $\max \Delta x_k$ is called the mesh size of the partition.

If the limit of the Riemann sums exists as $\max \Delta x_k \to 0$, this limit is known as the Riemann integral of $f(x)$ over the interval $[a, b]$. The shaded areas in the above plots show the lower and upper sums for a constant mesh size."[16] (Courtesy of: "Riemann Sum." *From Wolfram MathWorld*, mathworld.wolfram.com/RiemannSum.html.)

By using the Riemann sums we can see that the approximation of an integral gets closer and closer to the actual value of the integral of, in this case, a parabola. Now you might be asking that how is an integral something that we can calculate empirically. Well this method of drawing rectangles under the curve is something that before we use the actual calculations we do because it's the intuitive and empirical way of calculating the integral.

Now the derivative is something that is not so empirical but nevertheless the derivative as a concept is something that can be seen as quite empirical because of the fact that we are looking for the tangent in curve and we do this by approaching the points in two different places of the curve until the distance between the points is 0:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ [17]

[15] Images courtesy from: "Riemann Sum." *From Wolfram MathWorld*, mathworld.wolfram.com/RiemannSum.html.

[16] "Riemann Sum." *From Wolfram MathWorld*, mathworld.wolfram.com/RiemannSum.html.

[17] "Derivative." *Wikipedia*, Wikimedia Foundation, 4 Mar. 2019, en.wikipedia.org/wiki/Derivative.

The equation that we have here the h stands for the height difference between one point of the tangent going through the curve and the limit is just saying that it has to approach 0 to be the tangent.
This way of calculation tangents is one we can do by drawing and seeing it. This as well as the Riemann sums for integrals is a very visual way of calculating derivatives and thus deserves a mention in this paper due to its empirical nature.

Conclusion

After reading this paper we can see that the building blocks of basic and important branches of mathematics are facts that have an empirical nature. In arithmetic we can see in the way we: sum, subtract, multiply, divide and see numbers we do it empirically or the basis of that knowledge is empirical. Then in number theory we can see the use of that we call “Quasi-proves” for some conjectures. In Geometry we can see the generalization and idealization of physical objects in our minds to construct them word by word. Afterwards in Set theory we can see the importance of empiricism and dealing with sets of objects. In Functions we can see the importance of generalization and idealization of curves in the real world. And then in calculus we can see the empirical methods used to calculate both: the Integral and the Derivative.
As final conclusion, we have shown that the nature of mathematics is empirical by examining some of the basic fields within it.

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MP56. MATH AND PHYSICS IN BILLIARDS(POOL)

Dimitar Marinov Lazarov
125 SU "Boyan Penev", Bulgaria

ABSTRACT

In this paper I will write about math and physics in billiards. I am going to start with angles, then I will move on to the physics where I will calculate and explain forces and speed of a ball whilst including air resistance (drag). Then in the end I will try to simply explain the math when a ball is spinning sideways and moving forward.
I will work in general international measures for a ball and a table.

INTRODUCTION

In this paper I will be talking about what are the simplest parts of Billiards Mathematics and Physics. Those are angles – for the math part, and velocity – for the physics part. I find these to be the simplest and/or most interesting parts of the theme. I will also show these because they are some of the most important "beginner" things to know.
Just one more quick thing. In the physics part I will leave a few variables because those things depend on how hard you hit the ball and that can't be a precise measurement.

BASE

As a base I mean a quick explanation about what measurements I am going to use here. First of all I am going to use meter per second for speed, grams and kilograms for weight and all internationally accepted, most common sizes for the balls and for the table, I mean the size you will most likely find in a bar, a hotel , etc. The table have a few sizes but the exact size doesn't quite matter. I will use the ratio of the sizes, it is 2 to 1. I will count the ball as 165g and it's diameter-57.2mm. I will also look at the table as being 700kg (Some are more, some are less, this is kind of close to the average).

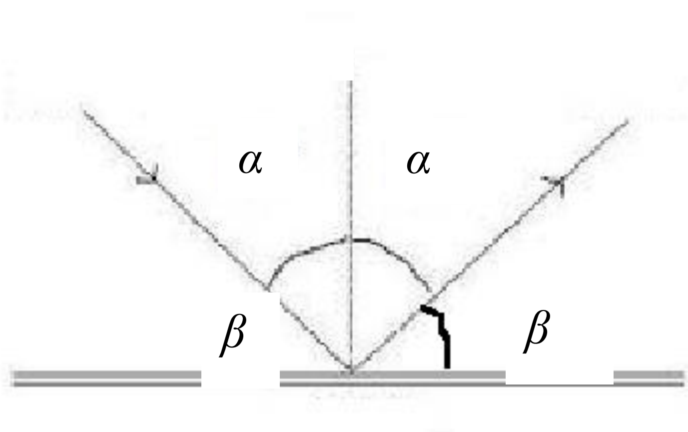
For the Mathematics part

Usually billiards is played on a rectangular table and there are 15 balls, but Billiards mathematics is a lot simpler (kind of). This is because Billiards mathematics is quite different from normal Billiards. In Mathematics or Geometry the sport is shown in two dimensions, and in it simplest goes like this:
Billiards mathematics looks into the path of a point (the ball) within a closed object (the table – not necessarily rectangular but in other shapes as well(ellipse, triangle, etc.)). This is the simplest version because it also goes into the path of the point inside a three-

dimensional object but I will not look into that for the sake of simplicity and conservation of time for the presentation.

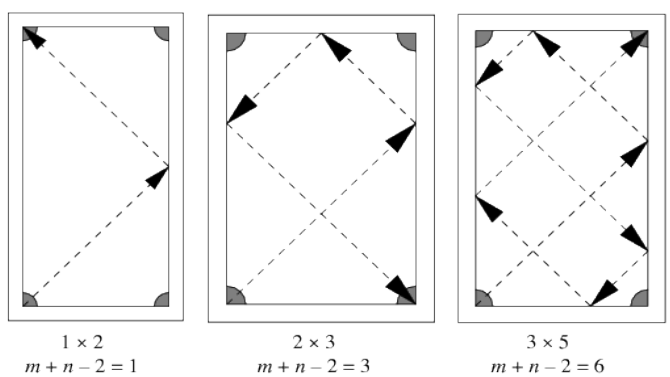
Angles

With the angles I will look first into tables with straight lines and then into elliptical tables, because in tables with curved walls the geometry is different.
With the balls in billiards is like with light in physics, everything is, if you are moving in a straight line and you hit an obstacle you will bounce off in a mirrored angle to the one you hit the obstacle at. What that means is that if a ball hits a wall in an angle α it will bounce off at an angle α. This is nicely shown with this image:

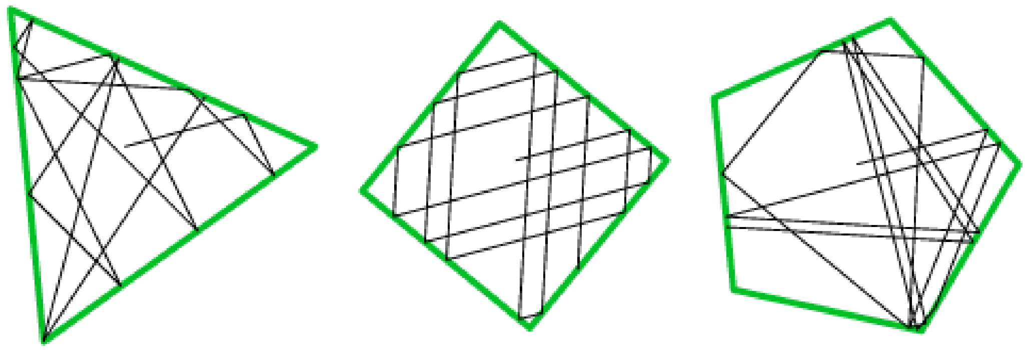


Where α+β=90o , the line with the arrows is the path of the ball and a is the wall of the table.

In the image on the right you can see the path of the ball when it hits the wall under a 45^0 angle with different sized tables (below each image you can see the ratio of the sizes). The ball starts in the bottom left corner.

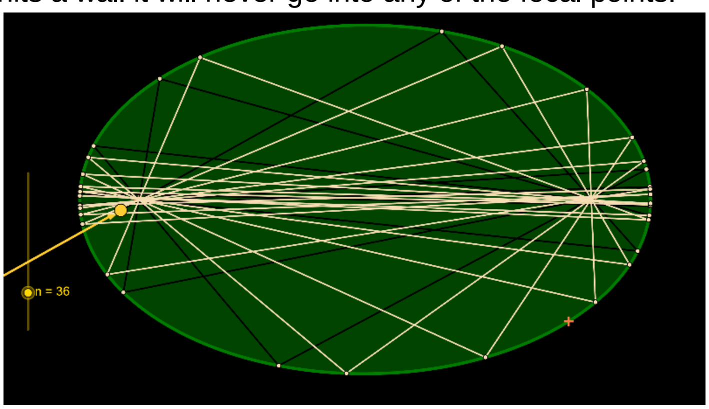


Now that was only for rectangular tables, here are examples for a triangular, a square and a pentagon table:

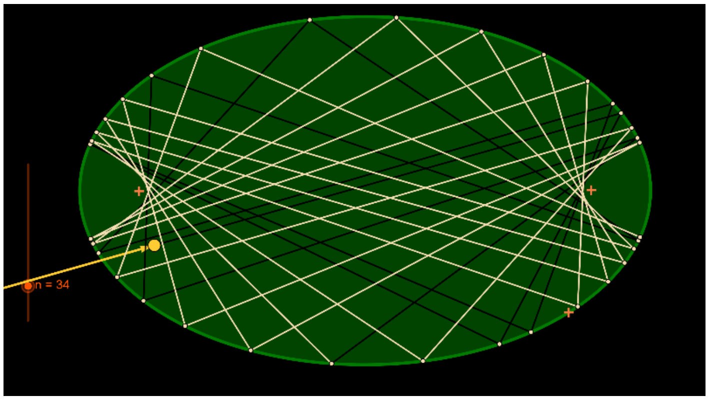


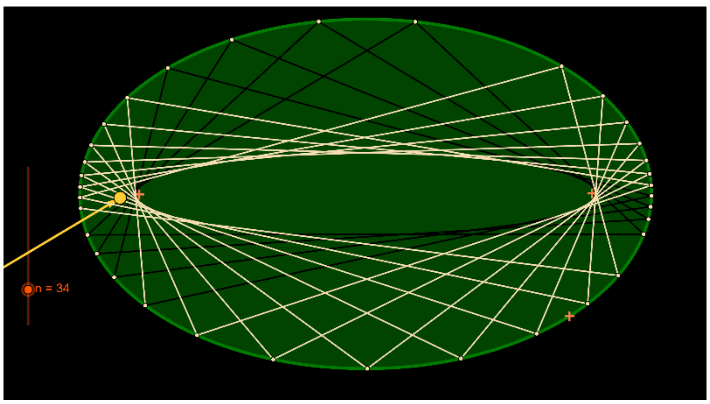
Elliptic Table Billiards

As I said that was on tables with straight lines so now let's look into elliptical tables shall we. Now first what is an ellipse? An ellipse is a geometrical set of points M for which the sum of the distances to two focal points F_1 and F_2 is equal. In other words – $|F_1M| + |F_2M| = C$ (C = constant), for any point M from the ellipse where F_1 and F_2 are the focal points.
In billiards if you hit a ball and it goes through one of the focal points of the ellipse it will bounce into the other one. And if the ball doesn't go through a focal point after it has been hit and before it hits a wall it will never go into any of the focal points.



In this image you see the path of a ball that has gone through a focal point.





In these images you can see two paths for the ball when it doesn't go through any focal points. It is quite fascinating how it gets so close to the focal points but never goes through isn't it.

Interesting Fact About Billiards on an Elliptic Table (Loop)

EIGHT BALL IN the corner pocket? Not on Alex Bellos' billiard table. It has no corner pockets—indeed, it has no corners whatsoever. Bellos, a British journalist who covers sports and mathematics, combined his two obsessions to create a unique game that he dubbed Loop.
While working on his book *The Grapes of Math*, Bellos became fascinated by the ellipse. "The shape has these wonderful geometrical properties that we've known about since the ancient Greeks," he says. Instructors frequently illustrate these properties by describing how balls would rebound on an elliptical pool table. Bellos decided to bring that thought experiment to life, refining a set of rules for a two-player game that takes full advantage of the shape.
Loop requires a cue ball, three coloured balls, and an elliptical table with one uncharitably small pocket. The billionaire quantitative investor David Harding—a noted benefactor of worthy scientific causes—underwrote the five-figure cost of a single luxuriously appointed Loop table. "If any American billiard companies want to produce more, I'd love to work with them.", Bellos says.

From "WIRED" post "ELLIPRICAL POOL: AN EVEN NERDIER SPORT THAN QUIDDITCH" (Also linked to in Sources)

Physics Part

Now for the part where I tell the physics.
At first, I am going to explain how much of it's speed a ball loses when it hits a wall. Let's say that at first the ball has a velocity of u_1 and the table u_2 before the collision and velocities v_1 and v_2 and the mass of the ball is m_1 and the mass of the table is m_2. I am talking about the table as well because a common misconception is that when a billiards ball hits one of the walls of the table, the ball continues moving with the same speed and the table doesn't move. That is not true however because the table moves. It absorbs some of the force of the moving ball and because this force has a direction the table moves forward a bit, but this movement is so little that it's invisible to the naked eye. So the ball loses some speed when it hits a wall.
This is called elastic collision. When it's perfect (And I am going to count it as if it is perfect) we can calculate v_1 and v_2 from where we can calculate the speed of a ball after it hits the wall and before it has hit it.

The conservation of the total momentum before and after the collision is expressed by:

$$m_1\overrightarrow{u1} + m_2\overrightarrow{u2} = m_1\overrightarrow{v1} + m_2\overrightarrow{v2}$$

Likewise, the conservation of the total kinetic energy is expressed by:

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_1^2$$

From these two we can get this equation for direct estimation of v_1 and v_2:

$$v_1 = \frac{m1-m2}{m1+m2}u_1 + \frac{2m2}{m1+m3}u_2$$
$$v_2 = \frac{2m1}{m1+m2}u_1 + \frac{m2-m1}{m1+m2}u_2$$

Now because in the beginning the table's velocity is 0 ($u_2 = 0$) we can compress the equations to:

$$v_1 = \frac{m1-m2}{m1+m2}u_1$$
$$v_2 = \frac{2m1}{m1+m2}u_1$$

We can calculate the masses of the objects by dividing their weight by the acceleration of gravity (which we will use as =10 for simplicity) and we get m1= 165g/10= 16,5g= 0,0165kg and m_2= 700kg/10= 70kg.
When we plug these numbers into our equation, we get this:

$$v_1 = \frac{-69{,}9835}{70{,}0165}u_1 = -0.99952868252483343211957181521499u_1$$
$$v_2 = \frac{0{,}0330}{70{,}0165}u_1 = 0.00047131747516656788042818478501496u_1$$

We are going to come back to this after we estimate u1 because for that we will need the speed of the ball and calculating that we can estimate how much distance it will travel over time (The velocity – u1).

The force of the ball (F) is equal to W*H, where W is the weight of an object and H is the acceleration it has been given by the hit from the stick, minus the drag force (D)
For billiards balls W=165 grams = 0,0165 kg

$$H \text{ is variable and } D=\frac{Cd*\rho*A*v}{2}$$

In this equation Cd is the drag coefficient, ρ is the density of the ball, A is the cross-sectional area of the ball and v is the speed of the ball relative to the air.
The Cd of a round smooth sphere in air is about 0.47.

$$\rho_{\text{cue ball}} = \frac{mass}{volume} = \frac{0{,}0165kg}{\frac{4}{3}\pi r^3} = \frac{0{,}0165kg}{\frac{4}{3}\pi.(57{,}2\text{^}3)} = \frac{0{,}0495}{4\pi.(57{,}2\text{^}3)} =$$

$$= 0{,}0495\ kg/4\pi.0{,}0572^3 m^3 \approx 0{,}0495:0{,}00078392893685554 \approx 63{,}143\ kg/m^3$$

$$A = r^2\pi = 0{,}0572^2\ \pi = 0.00327184\pi\ m^2$$

v is equal to the speed of the ball only, because the air is not moving until the ball passes through it, so we end up with this annoying equation:

$$F = 0{,}0165H - \frac{0{,}47.\frac{63{,}143kg}{m^3}.0{,}00327184\ \pi\ m^2.v}{2} = 0{,}0165H - \frac{0.47.68.148.0{,}00327184\pi\ .v}{2} = \quad =0{,}0165H - 7.7380324736\pi.v$$

Force is equal to mass times acceleration so to get the velocity we will calculate acceleration by dividing the equation above by the mass (0,0165 kg). So we get to this:

$$H - 468.971665066666666666666666667\pi.v \approx H\text{-}468{,}972\pi.v$$

So back to our v_1 and v_2

v_1 is now equal to:

$$-0.99952868252483343211957181521499.(H\text{-}468{,}972\pi.v) \approx$$
$$\approx -0{,}9995.(H\text{-}468{,}972\pi.v)\ m/s$$

and v_2 is equal to:

$$0.00047131747516656788042818478501496.(H\text{-}468{,}972\pi.v) \approx$$
$$\approx 0{,}0005.(H\text{-}468{,}972\pi.v)\ m/s$$

Because as we calculated $u_1 = H\text{-}468{,}972\pi.v$

CONSLUSION

Our structures on the Mathematical side of things show that there's always a pattern that things follow – here it is a mirrored angle or the fact that the ball will always go exactly to the other focal point if it passes through the first one.
And from the Physics part of things we understood that we can't precisely adjust everything, but we can always get pretty close to the truth.
And if a person wants to start out in billiards this is the way to go, this and practise. Although I play some billiards and even though I know all those formulas an stuff I'm still pretty bad, because I am not accurate in the first place.

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MP65. A DEDUCTIVE DATABASE APPROACH TO AUTOMATED GEOMETRY THEOREM PROVING AND DISCOVERING WITH JAVA GEOMETRY EXPERT

Kostas Georgios-Alexandros, Bampatsias Panagiotis
Varvakeion Model High School, Greece

ABSTRACT

The purpose of the present paper is to show that the development of the computer program JGeX enables one to solve many elementary and non-elementary problems of classical geometry. Classical methods show the beauty of geometry and provide better insight into the situation. However, computer methods allow one to solve complex elementary and non-elementary problems. JGeX is a software tool for dynamic geometric drawing and automated geometric theorem proving and discovering. The dynamic geometric editor has the following functions: (1) geometric input of geometric diagrams and statements for the prover, (2) dynamic geometric transformations and measurements, (3) animation and loci generation. JGeX is also an efficient computer program for geometric reasoning, implementing four proving methods, including algebraic ones. The prover of JGEX has been used by many researchers to deal with various kinds of geometry problems.

INTRODUCTION

Mathematical Proofs

The word “proof” has different meaning in different sciences. There is no accurate definition of proof in Mathematics. We could say that a mathematical proof is a procedure that we follow in order to solve a problem and to justify why a given solution to a problem is in fact a correct one. Mathematical proof is an invention of ancient Greek Mathematics. The first proofs of this kind are attributed to Thalis and Euclid, who set the foundations of modern geometry.

Formal and Informal Proofs

Mathematical proofs can be distinguished in two big categories, formal and informal.
Formal proofs refer to typical procedures and are an immediate result of logic rules. They are usually applied on axiom systems and are very useful in modern applications in Informatics. For example, Automatic Proof Systems (JGeX, Coq, Isabelle), Artificial Intelligence (AI) and Autonomous systems heavily depend on this kind of procedures.
On the other side, Informal proofs also utilize deductive rules. However, steps could be skipped and approaches could be generated ex nihilo. In those kind of proofs, we find ideas that can’t be extracted directly from a formal procedure. Typical examples of informal proofs are Euclid’s

proof concerning the number of prime numbers and Gauss' proof regarding the orthocenter which we are going to present next.
The proof of Gauss concerns the three altitudes of triangle ABC in figure 1. What we want to prove is that point H is the intersection of the three altitudes. The classical geometric proof of Gauss is based on the construction of the triangle HJI, where the lines HJ, JI, HI are parallel to BC, AB and AC. The points A, B, C are the midpoints of HJ, HI and JI. Then AD, EB and GC are the perp-bisect lines of HJ, HI, JI in triangle HJI, which concur.

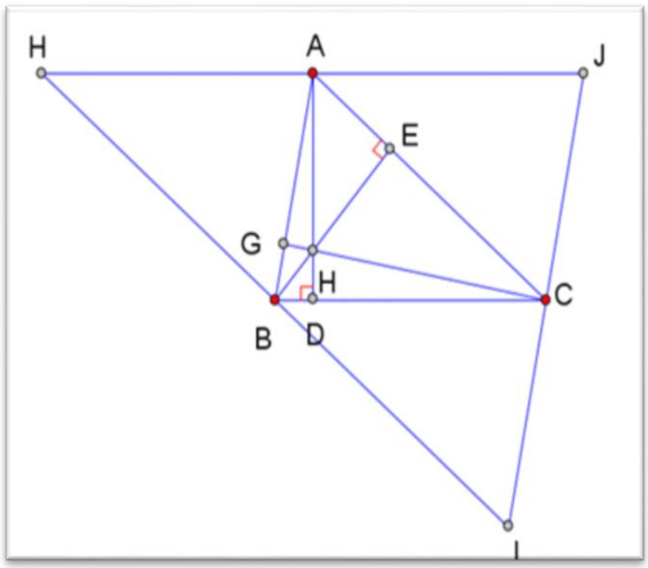

Figure 1: Classical Geometric Proof - Gauss

This classical geometric proof is based on the construction of three new points H, J and I, which can't be generated directly by a formal procedure and are based on the geometrical intuition of Gauss. No system is able to guess that constructing those 3 points would help facilitate the proving procedure.

Mechanical Proof Systems – JGeX

A Mechanical Proof System is software, which, under certain circumstances, is capable of performing formal proofs based on a collection of fundamental axioms. Some well-known systems of this kind are Geometrix and JGeX (Java Geometry Expert).

Figure 2 – Core of JGeX rules

Let's explain further what JGeX is, since this is the major concern of our project. JGeX is just like any other interactive dynamic geometry system including the additional capability of performing proofs if certain guidelines are followed. It was developed on 1980 by Shang Chou, Xiao Shan Gao and Zheng Ye, and it is considered one of the most complete environments in the field of Automatic Proofing. JGeX offers specific tools for designing geometric figures and enforces more formal design and construction rules. It contains a core of 43 rules (figure 2), most of which are common theorems of Euclid Geometry. Based on them the program is able to verify theorems by generating proofs. This environment features 4 different proving methods including the deductive database that we are going to use. There are also 2 algebraic really powerful proofing methods, Wu's method and Groebner basis. However, their examination is beyond the goals of this project. It's important to note that the combination of those different proving methods is what makes JGeX one of the most powerful Automatic Proof Systems that have ever been created.

Fixpoint and Proving Procedure

One of the most useful capabilities of this tool is Fixpoint, a library that contains all the geometric properties that can be extracted from a given scheme. It is the starting point of the Deductive database proving method and enables students and teachers to evaluate claims or extend a proof. This library appears as a window where all the properties that the program was able to extract are ordered according to specific categories as shown in figure 3. Clicking on any of those properties will highlight all involved geometric objects in the design window. It is a feature that can undeniably add a lot of educational value to the system, since it allows students to have a better understanding of the problem by experimenting with various properties.

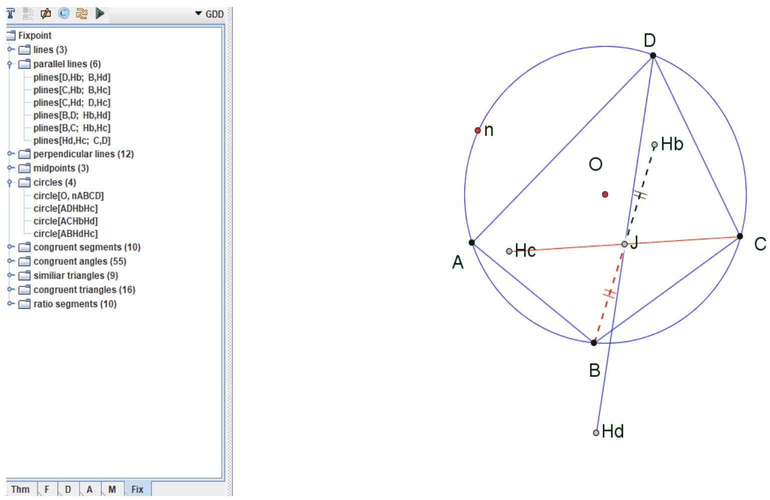

Figure 3: Fixpoint Library for a geometry problem

JGeX contains a closed core of 43 rules, which includes some fundamental axioms and theorems. Unlike other similar system's cores, the one of JGeX is locked and can not be edited or extended. This is one of its main disadvantages. This list is, by no means, complete and that's why some theorems cannot be **directly** proved using this program. However, it is sometimes feasible to do a proof of our own based on the various geometric properties that Fixpoint enables us to examine. JGeX will start from this core of axioms and hierarchically "build" the property list of all the geometric items of the theorem. At the same time, it will find the path of properties that will lead to the theorem we want to prove. By definition this is a genuine formal proof.
Since the program can function doing **only** formal proofs, it's crucial that we express our theorem in an indirect formal language, suitable for mechanical processing. Therefore, the way we construct our schemes is often of paramount importance and may block or allow the system to do a proof. When we ask the system to prove a sentence, it will translate all available data from the given scheme into algebra and will provide a quick answer as to whether our assertion is true or not using one of the really fast algebraic proving methods mentioned above. Then we can ask the system to prove the theorem using our preferred proving method. Depending on the size and complexity of the problem, this choice may have minor, or significant impact on the proof. For example, it will take a considerable amount of time to generate the Fixpoint library for complex schemes with a lot of geometric items and properties. Also, although some auxiliary points can sometimes be generated by JGeX, the deductive database method might often fail to prove a theorem that an algebraic method can prove within fractions of a second. Algebraic methods are not perfect though. They often provide no insights as to how the proof was done and are really difficult to use unless you have previous specific knowledge about how those methods work under the hood. That being said, we should choose the proving method wisely according to our needs. If we want to quickly verify an assertion and don't care about how the assertion is proved to be true, then we can safely use one of the algebraic methods JGeX features. On the other side, supposing that we don't have a high-

level knowledge about algebraic methods and we want to explore the problem in depth by examining its properties, it would totally make sense to use the deductive database method.

EXAMPLES AND APPLICATIONS

Example 1 – Orthocenter

Let's start by examining the machine proof of the theorem explained previously, that the three altitudes of any triangle concur. The machine proof is totally different. As said before, the system is not able to guess that constructing any additional points would help facilitate the proving procedure. Supposing that point F is the intersection of the altitudes BD and CE, we need to prove that AF is the third altitude and therefore BC ⊥ AF (figure 4). As a result, the system tries to verify that AC ⊥ BD, which is true by the hypothesis, and the equality between the angles [AC, BD] and [BC, AF].
What we have to highlight is that the system's proof starts with the theorem that we want to prove and ends up with the sentences of the hypothesis. As we can see in figure 5, the theorem we want to prove is on the top of the diagram, the in-between sentences are the assumptions that support the proving procedure, whereas in the last line we can see the sentences of the hypothesis. From this diagram we can clearly understand that the program can not work based on assumptions. As a result, the proof consists only of formal sentences verified by formal procedures. This example shows the difference between formal and informal proofs.

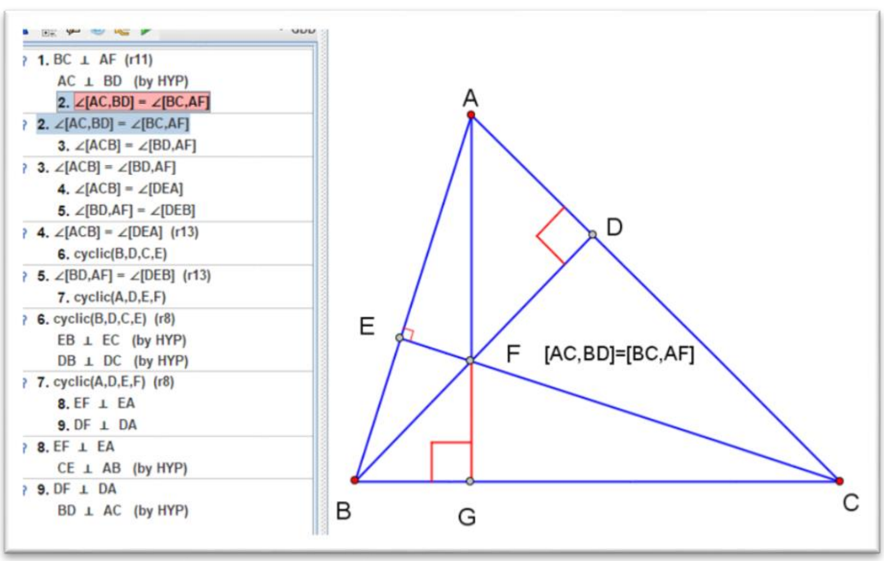

Figure 4: Orthocenter – Machine Proof

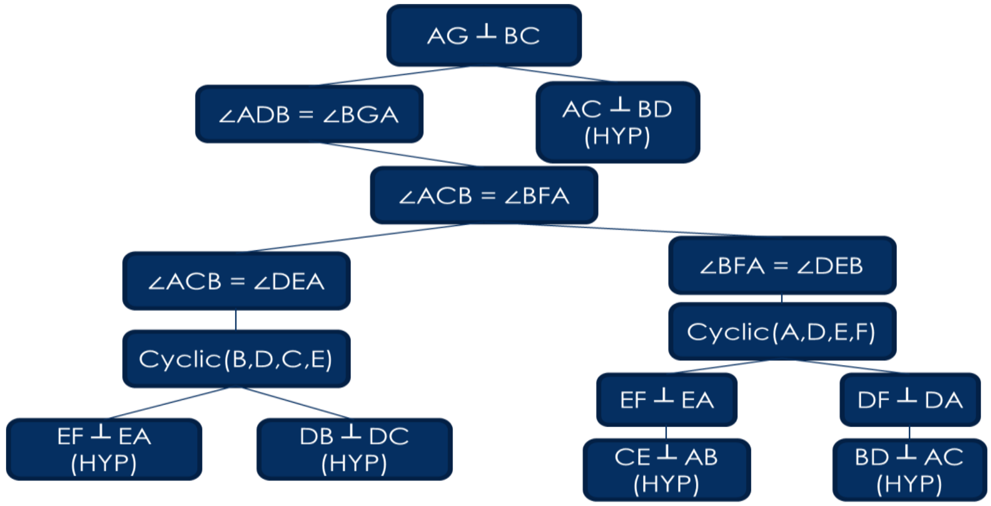

Figure 5: Orthocenter - Proof diagram

Example 2 – Mathematical Grammar School Cup

The following is a problem from the Mathematical Grammar School Cup in Belgrade in 2017. The hypothesis is:
*Let point **O** be the **circumcenter** of **triangle ABC** (figure 6) and let **D**, **E** and **F** be the midpoints of those arcs BC, AC, AB of **O**, that do not contain points A, B, C respectively. If P is the intersection of AB and DF and Q is the intersection of AC and DE, then **prove that PQ is parallel to BC.***

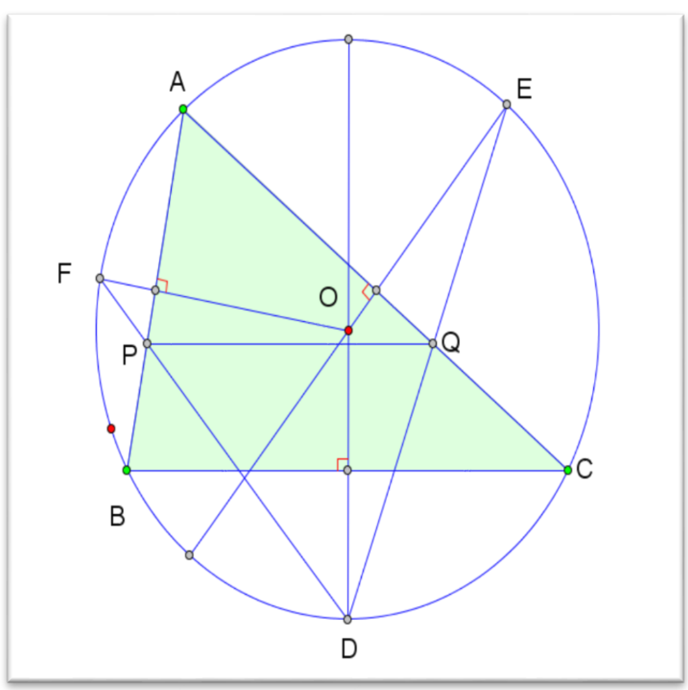

Figure 6: Problem from Mathematical Grammar School Cup – Belgrade 2017

The machine proves the theorem using only rules without adding any auxiliary point. The introduction of full-angles (see Appendix A – Full Angles) simplifies the predicate of the angle congruence. So the main goal is to prove the equality between the angles PQF and [BC, FQ]. The machine-proof is achieved in 39 steps, as shown in the left part of interactive window in figure 7, and the database contains 367 geometrical properties.

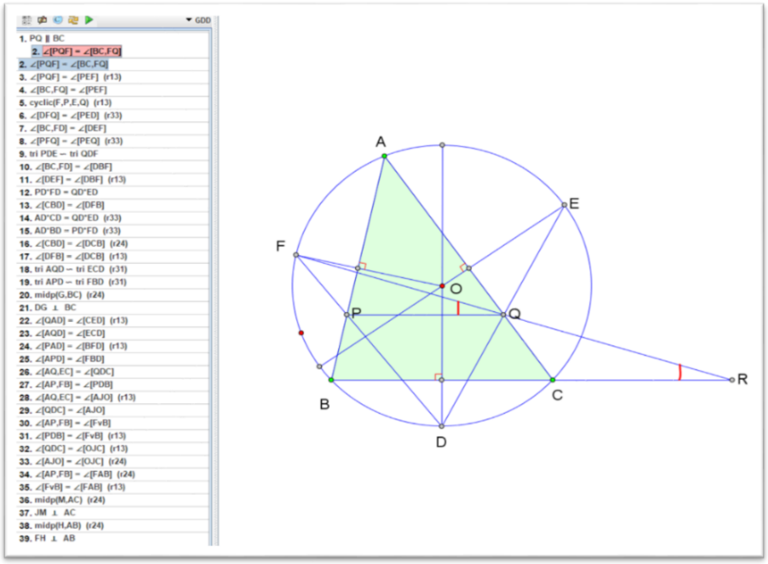

Figure 7: Grammar School Cup 2017 – Machine Proof

Example 3 – Mathematics Olympiad 1985

The next problem is from International Mathematics Olympiad in 1985. The hypothesis is: *Let A, C, K and N be four points* ***on a circle****. B is the* ***intersection*** *of AN and CK and M is* ***the intersection of the circumcircle of triangles*** *BKN and BAC.* ***Prove that BM is perpendicular to MO.***

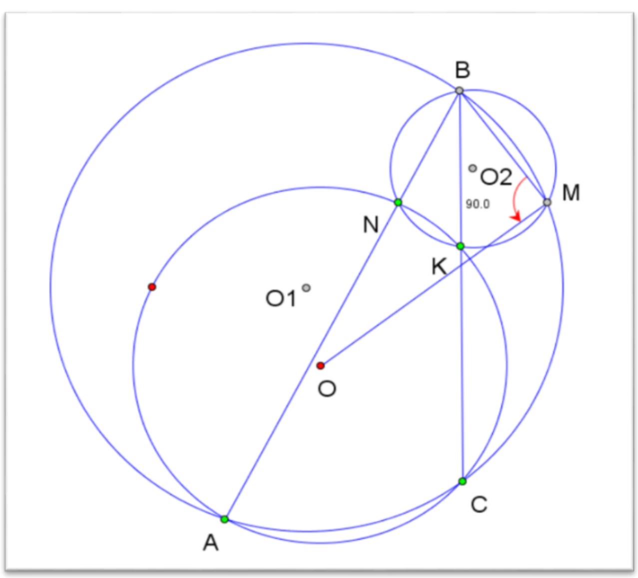

Figure 8: Problem from International Mathematics Olympiad 1985

It is important to mention that the machine's proof is based on the introduction of the auxiliary point D, the intersection of O1O and KN. The main goal of the system is to prove the equality between the angles [BMO] and [O1O2B] and to show that O1O2 is perpendicular to MB (figure 10).

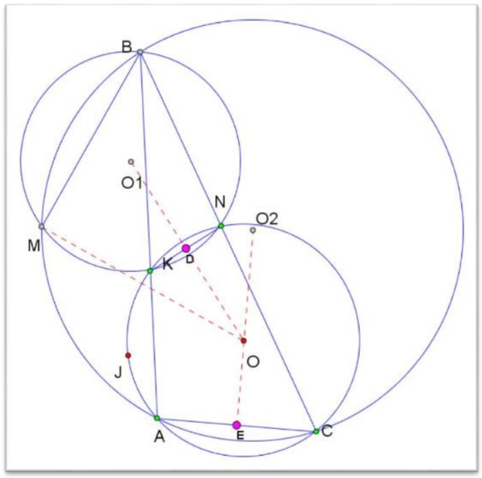

Figure 9: International Mathematics Olympiad 1985 – Machine Proof

CONCLUSION

JGeX is a system with a lot of capabilities:

- It is a particularly useful tool, maybe the best compared to other similar systems.
- This program can be utilized not only for performing geometric proofs, but also for the inductive approach of properties of geometric objects
- It enhances our intuition and creativity
- It helps us discover new geometric patterns and relations
- It allows us to try and verify assertions and analytically produced results
- It allows us to explore a result to verify whether an authentic proof is worth it

Of course, it has some drawbacks as well:

- The system is not easy to use by an amateur, which is reasonable given that it is not designed with education in mind
- The core of fundamental axioms is closed to editing, preventing us from understanding the impact that the fundamental theorems have in the proving procedure.

At this point, coming to the end of this paper, we would like to thank Mr Zenon Lygatsikas, our Mathematics Professor in Varvakeion Model High School for his guidance and assistance during the creation of this project as well as our families for their support.

APPENDIX A - FULL ANGLES

The full angle is one of the main geometric concepts of JGeX. The equality between two angles exists if these two angles concur with a rotation R. The introduction of full angles simplifies the predicate of the angle congruence. If using ordinary angles, we need to specify the relation among 8 angles and we need to use order relation to distinguish the cases.
A geometry predicate is for example the COLLINEARLITY of three points. There are several rules used in JGEX:
• If AB//BC then A, B, C are Collinear
• For points A, B, C, D, if AC⊥ BC and AD ⊥BD, then points A, B, C, D are cyclic
• One of the central geometric concepts is the full-angle. The full angle ∠ [u, v] is the angle from line u to line v. Note that u and v are not rays as in the definition for the ordinary angles
• Two full angles ∠ [[l, m] and ∠ [u, v] are equal if a rotation R exists such that R(l) // u Λ R(m) //v

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MP66. LINEARIZATION

Yavor Georgiev
125 SU "Boian Penev", Bulgaria

ABSTRACT

The presentation includes concepts of function, interpolation, approximation.
Presentation of the rounding method when calculating a function.
Explanation of interpolation and approximation.
Linearization as type of approximation.
Graphic representation of interpolation and linearization. Calculation of a linear function by tangent to a given point of a complex function.
First derivative of function.
Finding a square root by means of linearization. Problems and their solution.

1. CONCEPT OF FUNCTION

There are a lot of processes in a variety of industries that are described, programmed and controlled by features.
The function is a mathematical relation between a certain magnitude, which is called an argument, and another magnitude that is called value. Each argument matches exactly one value.

2. PRESENTATION OF THE ROUNDING METHOD WHEN CALCULATING A FUNCTION

Normally, when calculating the numerical values of some function f (x) at given points, we encounter the following difficulties:

The formula of the function is complex, e.g. $f(x) = \sin(\sqrt{\pi + \sqrt{x}})$

The results of the calculations are virtually always rounded, eg. $\frac{2}{7} = 0.28571 \ldots \approx 0.286$

Most real numbers are replaced with some rational, with a certain accuracy, eg. $\sqrt{2} \approx 1.4142$ or $\pi \approx 3.141593$.

3. CONCEPT OF INTERPOLATION

In the field of mathematics, interpolation is a method of constructing new data points within the scope of a discrete set of known data points.

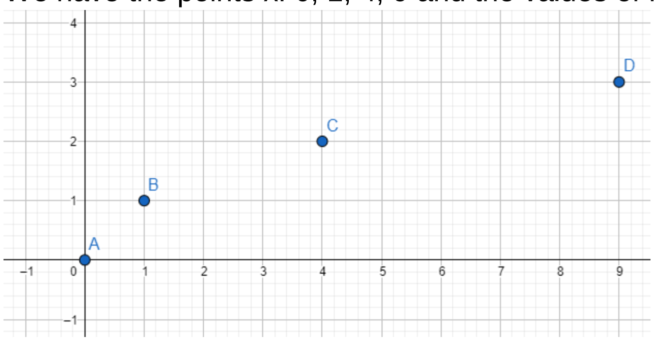
In engineering and science, there are often a number of data points obtained by sampling or experiments representing the function values for a limited number of values of the free variable. It is often required to interpolate, that is to say to evaluate the value of this variable's intermediate value function.

4. CONCEPT OF APPROXIMATION

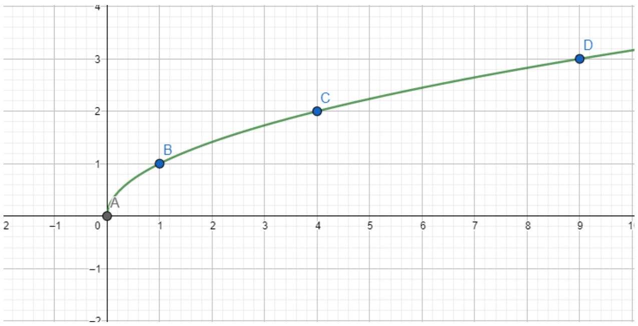
Approximation is the calculation of the values of the function with acceptable roundings. Linearization is a type of approximation.

5. EXPLANATION OF INTERPOLATION AND APPROXIMATION

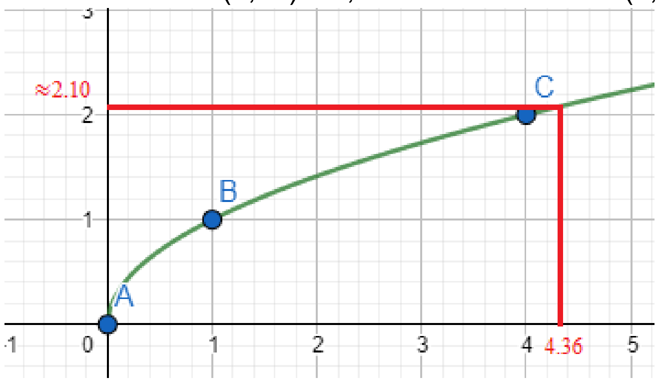
In interpolation, the f (x) function is unknown, but we have values for it at certain x values. Referring to these values we can with an acceptable rounding to determine the value of the function f (x) at a new value of x. Algebraically - by finding the function f (x) satisfying the given values. Graphically - by drawing an approximate graph of function by the known values. For a more accurate determination of the result, it is advisable to have more known points, possibly around the search value of x.
Example: We want to find through graphical interpolation f(4,36)
We have the points x: 0; 1; 4; 9 and the values of f (x): 0; 1; 2; 3



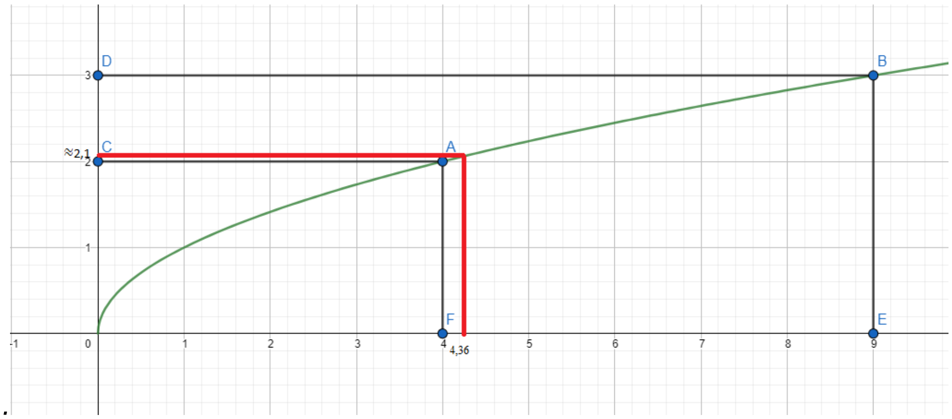
Drawing curve linking these points.



Determine that f (4,36) ≈ 2,1/ We assume that f (4,36) ≈ 2,1



In the approximation, the function f (x) is known and it is used when it is difficult to calculate the function values at each x



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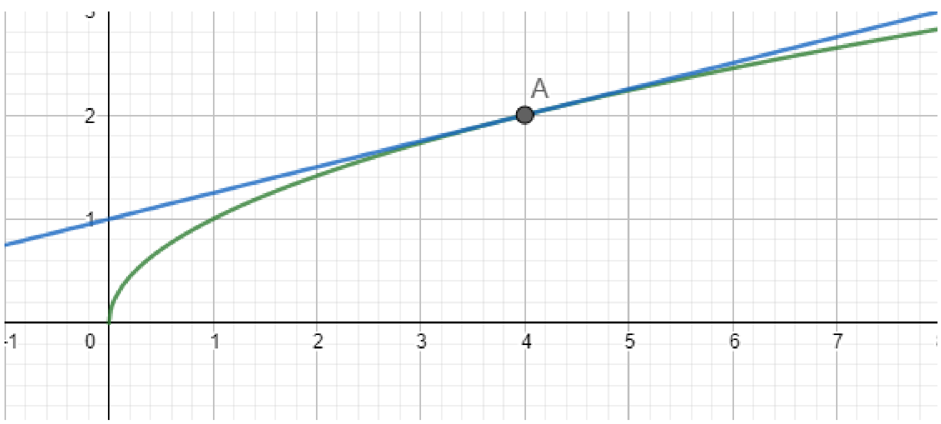
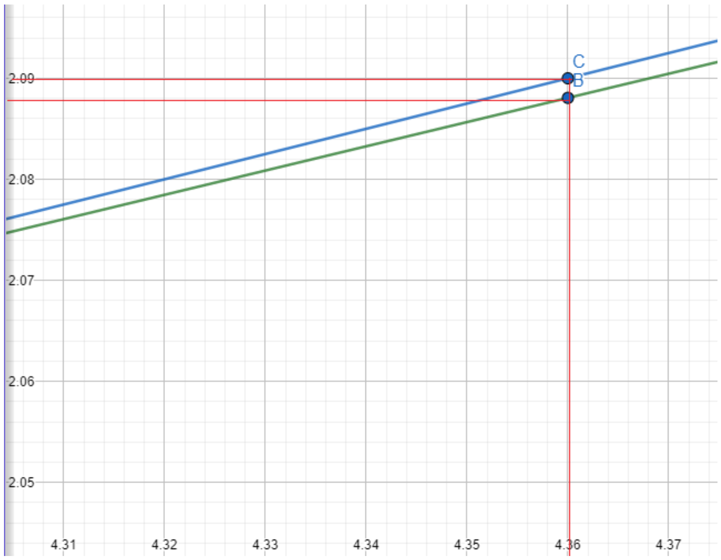
6. LINEARIZATION AS TYPE OF APROXIMATION

The idea is to match the complex function to a linear function within a sufficiently close range to the value looking for.
We want to find f (4,36), where f (x) = √x

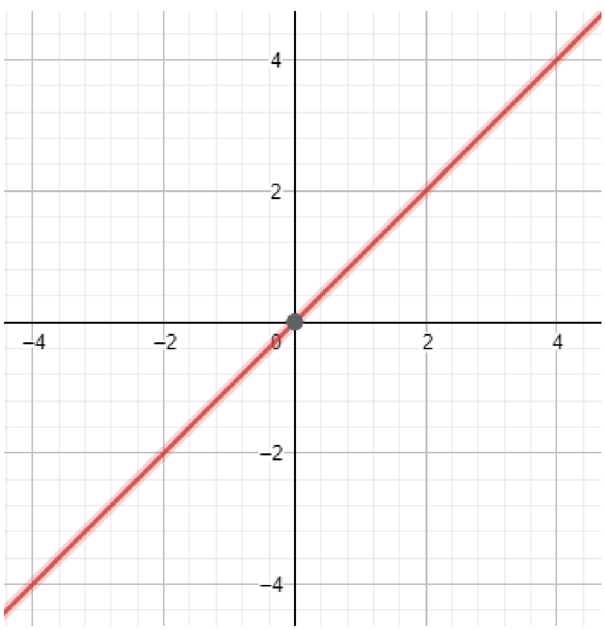


The closest value to 4.36 on the abscissa is point A.
We can easily calculate the value of C by the linear function (the blue line) and replace the hardly calculated value of the C of the complex function (the green line)

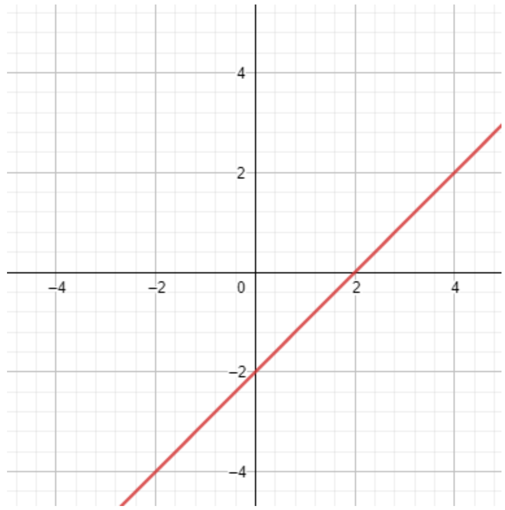


7. Calculation of a linear function by tangent to a given point of a complex function.

Let us take the function f (x) = x.



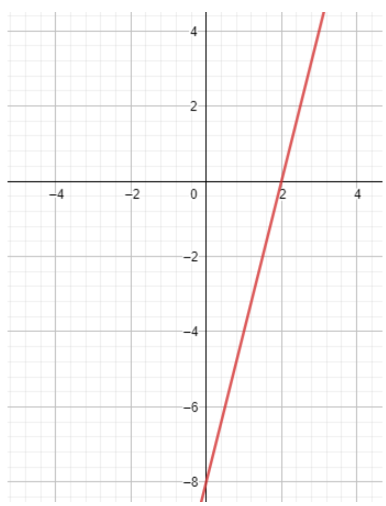
Let's move the red line two units to the right. A new function f1 (x) = x-2 is



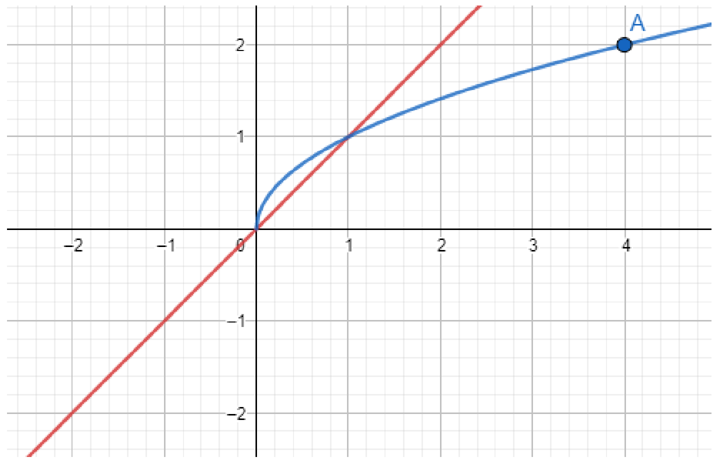
obtained. So, by adding and subtracting a number from the argument we can move the line along the abscissa and the ordinate (the line is moved 2 units down)

Let's tilt the red line. Again we get a new function

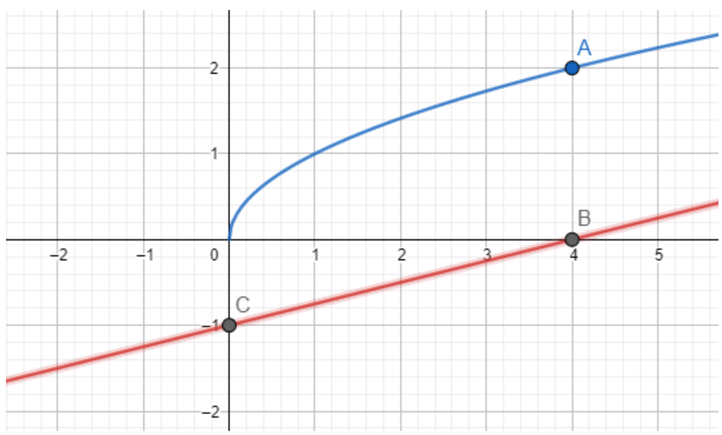
f2(x) = 4(x-2). Then to tilt the line we need to multiply the argument with a number.



Let us apply these rules in our case - we use the function f (x) = x. We need to move the red line to point A so that it is tangent to the blue line at that point.



Let's move it first on the abscissa. We get a new function - f1 (x) = x-4



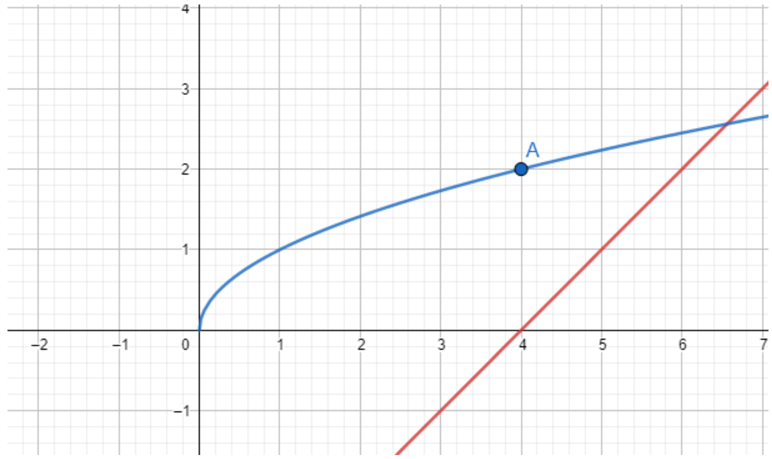
7.1 FIRST DERIVATE OF FUNCTION

To tilt the line so that it is tangent to f (x) = √x in point A, we need to calculate the first derivative of the function at this point. This is how the tilt angle coefficient is calculated at a given point in the function

f ’(x)=$\frac{1}{2}x-\frac{1}{2}=\frac{1}{2\sqrt{x}}$

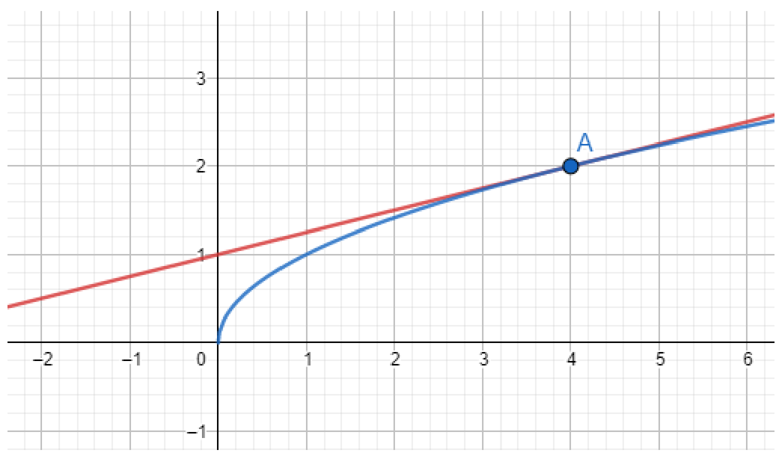
f '(4) = $\frac{1}{2\sqrt{4}}=\frac{1}{4}$

We get a new function - $f_2(x)=\frac{1}{4}(x-4)$. Now we only have to raise the red line to point A.



Again we get a new function $f_3(x)=\frac{1}{4}(x-4)+2$
We can now easily calculate an approximate value of f (x) = √x at x = 4.36

f3 (4.36) = $\frac{1}{4}$*(4.36-4) + 2 = 2.09*
Calculating with a calculator is 2.088.



Let us summarize the linear function for all complex functions F (x) in calculated point A with coordinates xA, yA. - f3 (x) = F '(xA) (x-xA) + yA
It is of utmost importance for the accuracy to define a close point to the point. Of course F(xA)=f3(xA)

8. PROBLEM 1

Find the value of f(x)=4x-$\sqrt{x}$ при x=5
Solution:
We know the value of the function at the close enough value of x=4.
f(4)=4*4-$\sqrt{4} = 14$
f'(x)=1-$\frac{1}{2}x - \frac{1}{2}$=1- $\frac{1}{2\sqrt{x}}$
f'(4)=1-$\frac{1}{2\sqrt{4}}$=1-$\frac{1}{8}$=$\frac{7}{8}$
F3(5)=$\frac{7}{8}$(5-4)+14=$\frac{7}{8} + 14 = 14\frac{7}{8}$
f(5) ≈ 14.8

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MP73.HOW TO WIN A GAME OF PAC-MAN?

Andrej Bozic, Luka Zmak
Prva Rijecka Hrvatska Gimnazija, Croatia

ABSTRACT

Games are used for fun but from every game you can learn a little bit of math. You might have played the game Pac-Man hoping to collect all the „Pac-dots“, sometimes you would succeed, but most of the time you would fail. When you lost you would blame it on your bad luck, but luck has nothing to do with this game because the movement of the ghosts who chase you is strictly defined and predictable. The ghosts' targeting schemes determine their movement and with the understanding of the schemes you can overmaster every level. Your next challenge was to complete it in a shorter period of time not repeating the path you once used, it was a hard challenge, but you could've solved it with math. Math gives you a new look on the game and it proves that for success you need logic and not luck.

SHORT HISTORY

The game was designed and developed by Toru Iwatani and a 9 man team, for the Japanese company NAMCO. The great success of the game can be witnessed form its 1 billion profit within a year and a grand total of 400,000 arcade machines were sold by 1982 worldwide. A perfect Pac Man game is where you score the maximum of 3,333,360 points and it was first achieved by Billy Mitchell on July 3rd, 1999. Unfortunately, he lost his record due to not using an original arcade machine. It’s interesting to mention that the last level (level 256) is bugged because the processor could register 255 numbers. The left half of the screen is displayed correctly but the right half is a mess of randomly coloured letters, numbers, and symbols.

Photo1 Original Pac-Man arcade machine

GAMEPLAY

You control Pac-Man which has a simple movement scheme of going up and down, left and right. The goal of the game is to collect dots (also known as Pac-dots) which give you points and you have to avoid the 4 deadly ghosts Blinky, Inky, Pinky and Clyde. Beside Pac-Dots points can be earned by collecting fruit that appears twice per level and 4 big flashing dots, best known as energizers which turn the tables on your enemies thus making them the prey. You have 3 lives and a life is lost when you get eaten by a ghost.

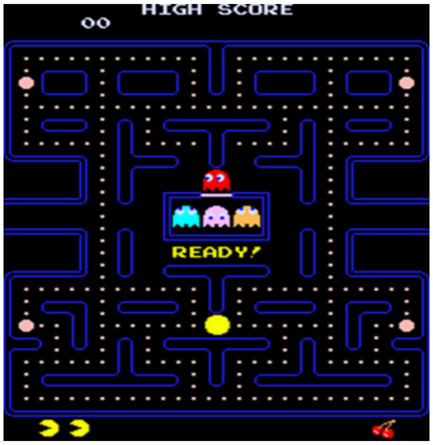


Photo2Pac-Man maze

THE GHOSTS

Blinky is the red and the most aggressive ghost. He has 2 modes the normal mode and the Cruise Elroy mode which is triggered by eating a certain amount of Pac-dots. In this mode his speed increases twice: once to match Pac-Man's speed and once more to go faster than Pac-Man. He targets Pac-Man's current position. Next up is Pinky as his name suggests he is pink. He always tries to go ahead of Pac-Man and tries to cut him off or ambush him. He targets 4 tiles in front of Pac-Man. An error occurs when Pac-Man is looking up that makes the targeted tile also have an offset of 4 to the left. Moving over to Inky he is quite unpredictable which makes him extremely dangerous. He has a complex targeting system. First he calculates 2 tiles in front of Pac-Man, then the distance from the new tile and Blinky, finally he doubles the distance between Blinky and the new tile in the direction which Blinky is facing. Lastly, there is Clyde he is the least threatening of the four ghosts. He targets Pac-Man's current position like Blinky but there's a catch that only happens if Clyde is located outside of an eight tile radius around Pac-Man, when he is inside the 8 tile radius he tries to find the quickest route to get out of it and after that the cycle just continues.

THE FASTEST WAY TO WIN A LEVEL

The fastest way would be to go over every path only once but unfortunately, that isn't possible. Let's start with a simpler example of a house. Can you think of a way where you go over a path only once, is it even possible? It actually is possible, there are quite a few ways you can do this and this is the method we chose, notice that the starting and the finishing point are different. Now let's get back to the Pac-Man maze, it is a lot more complicated than a simple house but WHY can't we go over every path only once? Is it because of the size or is it something else? The problem is connected to a much older problem from the 18th century.

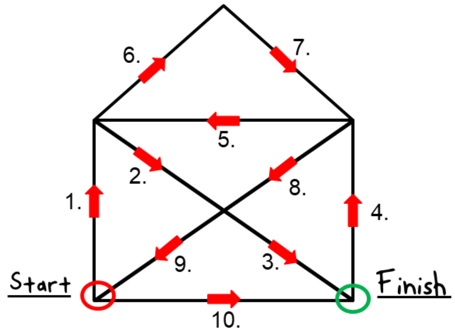


Photo3Fastest way to draw a house

SEVEN BRIDGES OF KÖNIGSBERG

It's The Seven Bridges of Königsberg problem. Königsberg (or Kaliningrad) was a city situated in north east Europe. The city had 7 bridges which connected the land over the Pregolya river. The townspeople of the city wondered while they were walking through the city could they go over every bridge exactly once. The problems negative solution was given by mathematician Leonhard Euler. At first he thought the problem had nothing to do with math but after further research he started to believe that this type of problems could belong to a new discipline of mathematics. It laid the foundations of topology and graph theory.

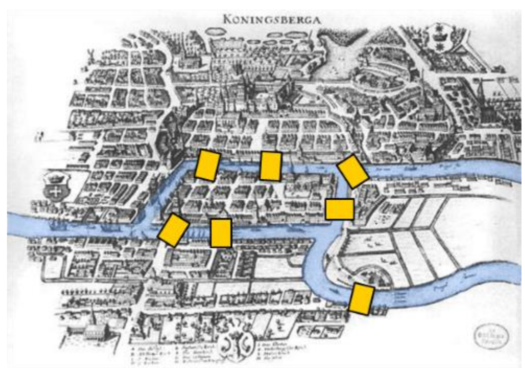


Photo4 Bridges in Königsberg

GRAPH THEORY

Graph is a geometric configuration which consists of vertices and edges, and an Eulerian path is a trail which visits every edge exactly once. Degree of a vertex is a number of pairings of a vertex. We distinct 2 types of Eulerian paths an Eulerian circuit and a semi-Eulerian path. An Eulerian circuit starts and ends on the same vertex and for that to be possible every node needs to have an even amount of edges connected to it. An example of an Eulerian circuit is the five-pointed star. Every vertex has 2 edges connected to it and as we determined that makes it an Eulerian circuit. A semi-Eulerian path must have 2 vertices with an odd number of edges connected to it and every other vertex must be an even degree. We can use the house example once again. The starting and the finishing vertex have an odd degree and every other vertex has an even degree and that's why it was possible to go over every edge once.

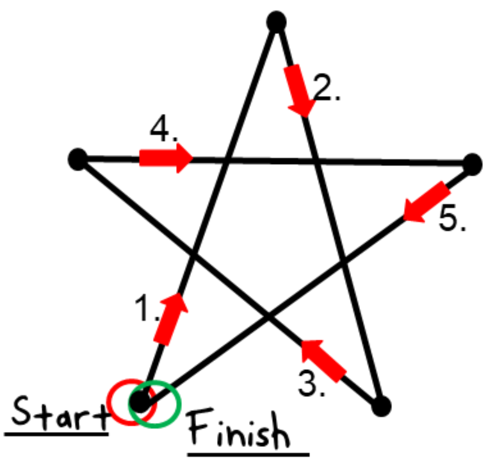


Photo6 Eulerian circuit

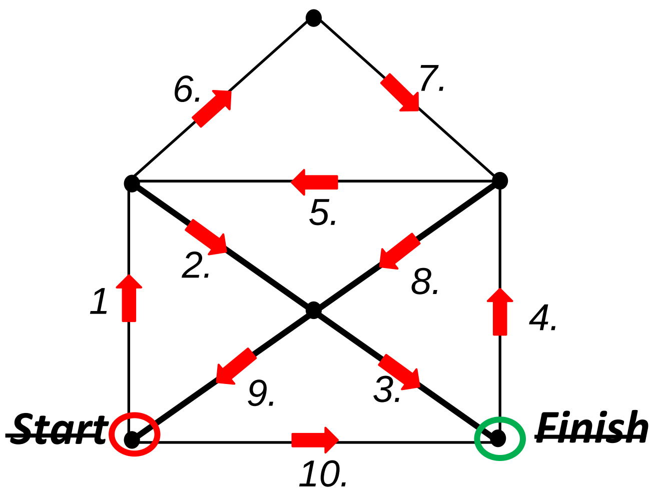


Photo5 Semi-Eulerian path

EULER'S SOLUTION

First Euler pointed out that the route and the starting point were irrelevant which allowed him to simplify the graph by eliminating the landmasses and the bridges by replacing them with vertices and edges. The vertices replaced the land masses and the edges replaced the bridges. Also, the length and form of the edges and the position of the vertices was irrelevant. The only important thing was the existence of connections between the vertices and the degree of the vertices. Euler came to the conclusion whenever one enters a vertex (land mass) by edge (bridge), one leaves the vertex (land mass) by edge (bridge) in other words one bridge is used to travel "towards" the land mass and one "away" from it, so the degree of the vertices must be even. In Euler's case all four vertices have an odd degree which meant that the Eulerian path wasn't possible and that meant the problem had no solution.

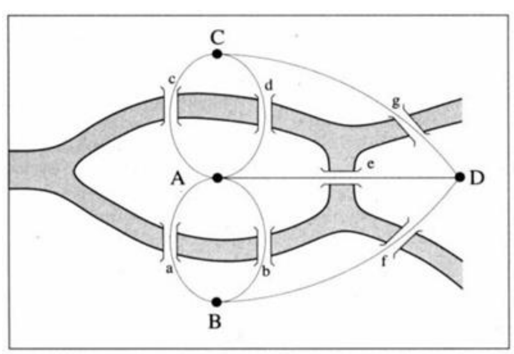


Photo7 Proof that problem doesn´t have solution

IN OUR CASE

The Pac-Man maze doesn't look so simple like the earlier examples and if you haven't guessed it already an Eulerian path of any kind isn't possible. The maze has a total of 20 odd vertices that are marked with red circles in the picture. The graph is a bit simpler than the actual maze because it only includes the edges and vertices which have Pac-dots on them. If we want to find the fastest way through the maze we will have to go over some edges twice and that means that we will need to connect the odd vertices with new edges. As you can see in the table below the number of pairings goes up quite fast and in our case we have 654 (six hundred and fifty-four) million 729 (seven hundred and twenty-nine) thousand and 75 (seventy-five) different pairing combinations. During our research we found out that the millions of combinations were put through a computer algorithm and the parts of the maze that you need to go over twice for the fastest way of completing the maze are in red in the picture on the left and in the picture on the right you can see an example of a possible path.

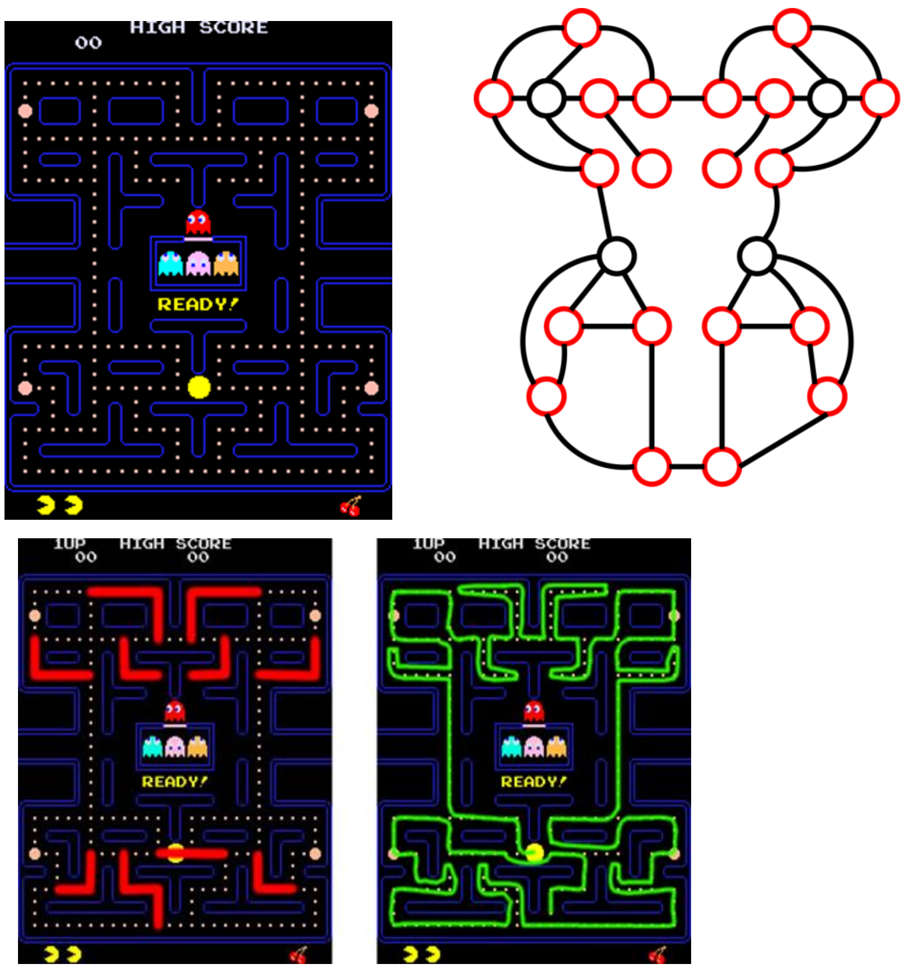


Photo8 Fastest way through the maze

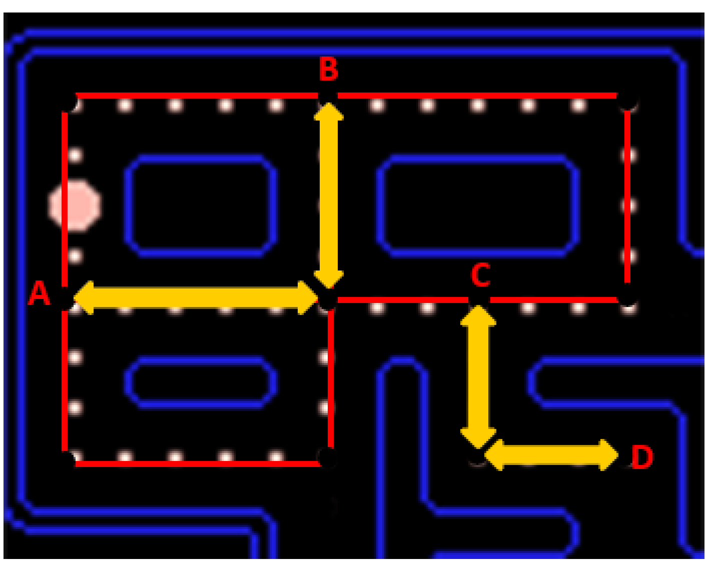
Number of odd nodes	Number of pairings
2	1
4	3
6	15
8	105
10	945
20	654,729,075

ODD NODES AND NUMBER OF PAIRINGS

Let's analyse the table and how the number of pairings grows really fast by increasing the number of odd nodes by a bit. The number of pairings is calculated by the formula (n-1)!!, where n is the number of odd nodes. First, we should explain what a factorial is. It's the product of all positive integers less than or equal to n. We mark it with a number and an exclamation mark beside it. The double factorial is quite similar, it's is the product of all positive integers from n to 1 that have the same parity as n meaning that if n is an odd number all the integers from n to 1 also have to be odd and vice versa for even numbers.

SIMPLER EXAMPLE

Let's simplify the maze by looking at the top left corner of the maze. We will use Pac-dots as the length of each edge. The first step is to determine the odd vertices and check what type of a graph it is. So, we have 4 odd vertices that we marked with A, B, C and D and that makes the graph non-Eulerian. The 2nd step is to calculate the shortest path between parings of the odd vertices. We have 3 combinations AB + CD which is 7+6 which is 13, AC + BD 7 + 12 which is 19 and AD + BC which is again 19. The shortest path is between parings AB + CD and those are the edges that you need to go over twice.



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MP74. MATHEMATICS IN CRYPTOGRAPHY

Nickolas Farmakis
American Academy Nicosia, Cyprus

ABSTRACT

Undoubtedly, cryptography is an essential tool for modern life. In today's world, there are countless examples of areas where cryptography is vitally needed. Blockchain, internet security and internet shopping are only a few examples of where cryptography is necessary. However, does cryptography relate to mathematics? The answer is an affirmative YES! Cryptography is a rigorous science that deals with the security of the messages that are being sent and received by two parties. That is, the fundamental problem in cryptography is that at any given time, a third party should not be able to eavesdrop or manipulate a message transmitted by one party to another party. Mathematics is the tool that can be used to solve this problem, since many areas of mathematics, such as number theory, matrices and algorithms are used in cryptography. For instance, many theorems involving prime numbers are frequently applied to a section of cryptography, named public-key cryptography, which will be analyzed in this article. In this paper, cryptography will be defined, but also briefly analyzed and mathematical definitions, as well as theorems will be observed and applied to cryptography. Consequently, a clear image will be drawn into the relationship between cryptography and mathematics.

INTRODUCTION

In order to understand cryptography, a sound mathematical foundation is required. Cryptography revolves mainly around the notion of encryption, which refers to an algorithm that takes some plaintext and changes it to a ciphertext using an encryption key, so that an attacker cannot decrypt the message without the encryption key. Moreover, encryption is divided into two sections; symmetric and asymmetric encryption, or else known as public-key cryptography, which will be seen in the paper.

With that in mind, mathematics offers numerous applications in encryption, such as number theory, matrices, as well as algorithms. Firstly, number theory is the study of integers and their properties, including divisibility, modular arithmetic and prime numbers. In parallel, matrices in mathematics are arrays of numbers or symbols, meaning they are numbers or symbols arranged in some order. Lastly, an algorithm is a set of instructions, including mathematical operations in order to carry out a specific function.

FUNDAMENTAL PRINCIPLES OF CRYPTOGRAPHY

Cryptography revolves around the notion of encryption. Encryption is responsible for turning readable data to unreadable characters that can only be turned back to readable data through decryption. Encryption is divided into two categories; symmetric encryption and asymmetric encryption, also known as public-key.

To begin with, symmetric encryption is encryption where the sender and the receiver have the same key that is secret and is used for encrypting and decrypting messages. In order to encrypt a message using symmetric encryption, the key and the message, also known as plaintext go through an encryption algorithm to produce unreadable data, also known as ciphertext.

To decrypt the message, the same key and the ciphertext pass through the decryption algorithm in order to generate the readable message back.
Public key cryptography involves a public key, that is known to everyone and a private key that is known only to the user. To encrypt a message, the sender has to encrypt using the public key and to decrypt the message, the receiver decrypts using his private key.

NUMBER THEORY IN CRYPTOGRAPHY

Number theory is the study of the integers as well as their properties.

Firstly, it is important to note what the greatest common divisor is, since it is extensively used in cryptographic protocols. To be more precise, the greatest common divisor, or gcd for short, is the largest number that can divide into two numbers. That is, the gcd of two integers a, b is an integer d such that:$d = ax + by$ for any integers x, y.
The gcd of a, b is written as (a, b). In addition, two integers a, b are relatively prime if $(a, b) = 1$, which means their gcd is 1.
An easy way to compute the greatest common divisor of two numbers is to use Euclid's algorithm. More precisely, Euclid's algorithm states that:
$(a, b) = (b, a - cb)$, for $a, b, c \in \mathbb{Z}$.
Euclid's algorithm is a very efficient algorithm in terms of time, since it can be executed by computers in a very small amount of time.

Furthermore, another significant tool in number theory is modular arithmetic. More specifically, modular arithmetic involves studying the remainders of numbers when divided by integers.
The modulo operator provides important information about the remainder of an integer when divided by an integer. That is, provided $|a| > |n|$, for any division $\frac{a}{n}$, there exists an integer q, such that $a = qn + r$, where q is the quotient and r is the remainder of the division. The remainder of the division can be denoted using the modulo operator as follows:
$r \equiv a \bmod n$
When two numbers have the same remainder when divided by the same number, they are called congruent. That is, two integers a, b are congruent if and only if:
$a \bmod n \equiv b \bmod n$, which is written as $a \equiv b \bmod n$
In modular arithmetic, the set of possible values for $a \bmod n$ is reduced down to $Z_n = \{0, 1, 2, \dots, n-1\}$, which makes the set of possible values much smaller in relation to the set of all the integers. This set is called the set of residues modulo n.
In this set, Z_n, there are two important inverses that must be noted:

1. Additive inverse: an additive inverse of x modulo n is a number y, such that $x \, mod \, n + y \, mod \, n \equiv 0 \, mod \, n$
2. Multiplicative inverse: a multiplicative inverse of x modulo n is a number x^{-1}, such that $x \, mod \, n * x^{-1} \, mod \, n \equiv 1 \, mod \, n$. To be more precise, an integer x has an inverse in Z_n if and only if gcd$(x, n) = 1$, for $x > 0$. This means that x and n must be relatively prime.

Moreover, a very famous theorem in number theory that has numerous applications in cryptography is Fermat's Little Theorem. This theorem states that for any integer $a,$ and prime number p:
$a^{p-1} \equiv 1 \, mod \, p$ if $a \not\equiv 0 \, mod \, p$
$a^{p-1} \equiv 0 \, mod \, p$ if $a \equiv 0 \, mod \, p$
This theorem is fundamental in public key cryptography, as it can be used for computing large powers of integers modulo prime numbers. This is extremely important, since prime numbers offer numerous applications in asymmetric encryption.
Furthermore, Euler's theorem has numerous uses, not only in number theory, but also in cryptography. To begin with, Euler's totient function, $\phi(n)$, is defined as the number of integers that are positive and are relatively prime to n. This means that $\phi(n)$ counts the number of multiplicative inverses in the set Z_n. In addition, it is important to mention that if n is a prime number, $\phi(n)$ is equal to $n - 1$, as all numbers less than a prime number are relatively prime to it. With these comments in mind, Euler formulated his theorem, which states that:
For $x, n \in \mathbb{Z}$, $n > 0$, $x^{\phi(n)} \equiv 1 \, mod \, n$
To add to this, the order of an integer a between 1 and $p - 1$, where p is a prime number is the smallest positive integer $k > 1$ such that $a^k \equiv 1 \, mod \, p$.
Parallelly, a generator is an integer g, $g \in Z_p$, with order $p - 1$. It is important to note that if p is prime, then Z_p has $\phi(p - 1)$ generators.

Now that several sections of number theory has been established, one can see how modular arithmetic applies to the One Time Pad, the only perfectly secure way to communicate, that is however not used in practice due to the fact that the key must be as long as the message, which is greatly impractical. In essence, the message, m, is added to the key, k modulo 2 and the result is the ciphertext, c.
$c = (m + k) \, mod \, 2$
To get the message back, the ciphertext c must be added to the key, k modulo 2.
$m = (c + k) \, mod \, 2$

Additionally, number theory can be observed in the RSA (Rivest Shamir Adleman) Algorithm, which is an asymmetric algorithm that uses numerous principles of number theory. This algorithm is secure and it has been used in numerous cryptosystems. In essence, as in every protocol in public-key cryptography, two sets of keys are needed; a public key and a private key. In the RSA Algorithm, the two sets of keys are large prime numbers. These prime numbers are denoted by p, q and their product, $x = pq$, but knowing that $\phi(x) = (p - 1)(q - 1)$, the two keys for encryption, k_1, k_2 are chosen such that:
$\gcd(k_1, \phi(x)) = 1$ and $k_1 \times k_2 \equiv 1 \, mod \, \phi(x)$.

This means that k_2 is the multiplicative inverse of k_1 in $Z_{\phi(x)}$. In this way, the public key consists of k_1 and x, while the private key consists of k_2.
To encrypt a plaintext m to form a ciphertext c, the following operation is done:
$c \longleftarrow m^{k_1} \bmod x$
In order to decrypt ciphertext c back to plaintext m the following operation is done:
$m \longleftarrow c^{k_2} \bmod x$
The security of the RSA Algorithm greatly relies upon the foundation that is very difficult to factor large numbers into their prime factors by brute force.

Consequently, the importance of number theory in cryptography has been observed, after close analysis of theorems in number theory.

MATRICES IN CRYPTOGRAPHY

Matrices are a very significant area in mathematics, but matrices parallelly have several applications in cryptographic protocols.

Firstly, it is important to define what a matrix is, as well as mention some definitions related to matrices. Essentially, matrices are arrays, or else a series of numbers and they can take many dimensions. A simple matrix is a two-dimensional matrix, such as the one shown below:
$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
Moreover, the determinant of such a matrix M is denoted by:
$Det\, M = |M| = ad - bc$
To add to this, the inverse of a matrix M is denoted by M^{-1} where:
$M^{-1} = \frac{1}{|M|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
From the above expression, two observations arise:
1. If $|M| = 0$, then M is a singular matrix, so that an inverse does not exist.
2. If $|M| \neq 0$, then M is a non-singular matrix, so that an inverse exists.

Matrices can be used to understand how encryption and decryption works. In essence, the key used to encrypt plaintext is a key matrix. This occurs through the plaintext being divided into different parts and each part being multiplied by the key matrix. Moreover, the ciphertext can be decrypted using the inverse of the key matrix. It is also significant to add that the more dimensions the key matrix has, the more complicated encryption and decryption become, thus the harder it is for eavesdroppers to manipulate or decode the ciphertext.

To illustrate how the key matrix can be used in encryption, let a two-dimensional key matrix, $K = \begin{pmatrix} 1 & 3 \\ 8 & 7 \end{pmatrix}$. Also, let the plaintext, $P =$ CRYPTO and let each letter of the alphabet correspond to the integer corresponding to their order, as in $A = 1$, $B = 2$ … $Z = 26$. Then $P = 3, 18, 25, 16, 20, 15 = \begin{pmatrix} 3 \\ 18 \end{pmatrix}, \begin{pmatrix} 25 \\ 16 \end{pmatrix}, \begin{pmatrix} 20 \\ 15 \end{pmatrix}$
Multiplying K with each separate part of P and taking the results modulo 26, the ciphertext, C is:

$$\begin{pmatrix}1 & 3\\8 & 7\end{pmatrix}\begin{pmatrix}3\\18\end{pmatrix} mod\ 26 = \begin{pmatrix}21\\150\end{pmatrix} mod\ 26 = \begin{pmatrix}21\\20\end{pmatrix} = T, U$$
$$\begin{pmatrix}1 & 3\\8 & 7\end{pmatrix}\begin{pmatrix}25\\16\end{pmatrix} mod\ 26 = \begin{pmatrix}41\\312\end{pmatrix} mod\ 26 = \begin{pmatrix}15\\0\end{pmatrix} = O, Z$$
$$\begin{pmatrix}1 & 3\\8 & 7\end{pmatrix}\begin{pmatrix}20\\15\end{pmatrix} mod\ 26 = \begin{pmatrix}35\\265\end{pmatrix} mod\ 26 = \begin{pmatrix}9\\5\end{pmatrix} = I, E$$

Therefore, C = TUOZIE.

This shows that matrices make encryption and decryption much more complicated, thus it is harder for eavesdroppers to intercept the key and read the plaintext.

ALGORITHMS IN CRYPTOGRAPHY

An algorithm is a set of mathematical instructions that can be executed in order to solve a specific problem. The three basic algorithms in cryptography are symmetric algorithms, asymmetric algorithms and hash functions. All of these algorithms use numerous mathematical principles, showing the significance of mathematics in cryptographic protocols once more.

Symmetric algorithms refer to symmetric encryption, which was analyzed before in this paper. Essentially, symmetric algorithms use the same key for encryption and decryption of data, thus if an eavesdropper intercepts the key, security fails. This also points to the key distribution problem, as the key for encryption and decryption must be transmitted between two parties that want to securely communicate. However it is insecure to send the key through the internet, as it can be intercepted by third parties, which means that these third parties can modify or read the messages. This is where asymmetric algorithms are able to solve this problem. The only symmetric algorithm still considered secure today is AES. Symmetric encryption and decryption can be summarized by the figure below:

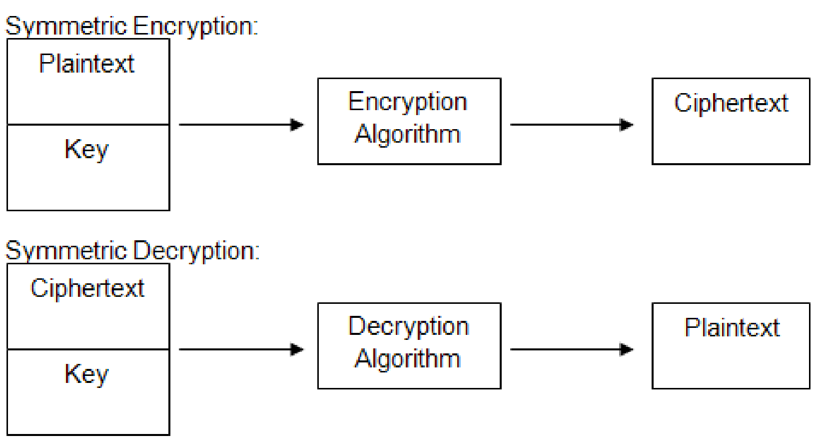

Figure 1

Asymmetric algorithms refer to asymmetric encryption, or else public-key cryptography. This is where every party has two sets of key; a public key, that is known to everyone, including third parties, and a private key that is unique to each party. Plaintext is encrypted through the public key, while it is decrypted using the user's private key. It is also possible to encrypt plaintext using

the private key and decrypting using the public key, which can be used to verify digital signatures. This solves the key distribution problem, as there is no need to transmit the private key between the two parties.
Some examples of asymmetric algorithms are RSA, which has already been encountered and Diffie-Helman, which is no longer considered secure. The following figure summarizes asymmetric encryption and decryption:

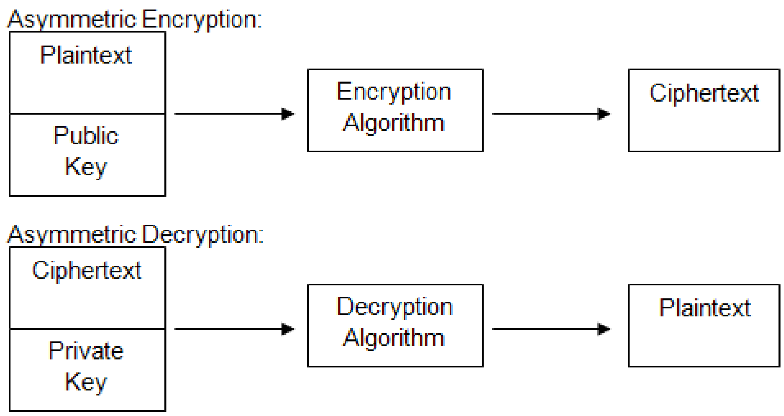

Figure 2

Additionally, hash functions are functions that take plaintext and compute hash values from that plaintext. Hash functions have no key, which is the reason that they are not reversible, thus the plaintext cannot be recovered from the hash value. The reason for the effectiveness of hash functions is the fact that it is extremely unlikely to have two plaintexts that output the same hash value. Several examples of hash functions include SHA-256 and MD5, which are considered secure.

To add to this, it is possible to combine some of these cryptographic algorithms to make a cryptosystem. More analytically, a cryptosystem is a cryptographic protocol that ensures both confidentiality and authenticity. Confidentiality is achieved through a form of encryption, such as asymmetric encryption, while authenticity is managed with the aid of a hash function. As a result, a combination of algorithms in a cryptosystem provides true security and this is the model used in digital encryption.

APPLICATIONS OF CRYPTOGRAPHY

Cryptography in modern society has numerous applications that are integral part of everyday life for people today. More specifically, cryptography is extensively used in the fields of cyber security, which involves internet security and the ability to have private accounts, as well as private conversations without third parties intercepting this information. It is also worthy to mention that cyber security today is extremely important, since people have the need of protection of personal data more than ever before. Moreover, cryptography is used in blockchain, as well as money transfers over the internet to ensure that money transfers occur securely and correctly.

Internet security revolves around the idea that a user should be able to safely access the internet without putting to risk his/her personal data. To achieve this, every web browser provides the user with some sort of encryption to protect him/her from malicious third parties that want to access his personal data. A cryptosystem can be used for this task, since, as mentioned before, a cryptosystem provides authenticity and confidentiality, thus the user is safe from hackers.

In addition, blockchain uses many cryptographic principles for its operation. Blockchain is defined as the technology that allows growing blocks of data, including cryptocurrency, such as bitcoin, to be linked together using cryptographic protocols. Cryptocurrency is any digital asset that can be exchanged for goods just like regular money. In blockchain technology, cryptographic algorithms, like asymmetric encryption and hash functions, are used to ensure that data remains secure and cryptocurrency transfers occur securely.

CONCLUSION

In conclusion, it is clearly evident that mathematics can offer numerous solutions to cryptographic problems. This proves how significant mathematics is for life today, as cryptography has fundamentally changed the way people live over the past few years, but without mathematical foundations, cryptography would not exist. Consequently, by encouraging mathematics, more cryptographic problems can be solved, leading to advances in the field of cyber security.

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MP76. MATHEMATICS AND DANCE: A UNIVERSAL LANGUAGE

Mia Cimas
Prva riječka hrvatska gimnazija, Frana Kurelca 1, Croatia

ABSTRACT

Mathematics is a language composed of symbols that are manipulated in such a way as to communicate certain relationships. Dance is also a language composed of relationships that are manipulated in such a way as to communicate certain symbolics. This presentation focuses on connecting the two through: dance canons and vectors, paths, sequences and order, focus and accent in dance, described by mathematical statistics and principles of interpolation and extrapolation and dance group formations and dance moves striving to describe geometrical shapes and figures. Each of these components is easily described by mathematical principles since dance is (like the universe itself) defined by space, time and energy. It is, just like mathematics, a universal language.

INTRODUCTION

"Mathematics is a language with which God has written the universe", said Galileo Galilei.
"Dance is the hidden language of the soul", said Martha Graham, the dancer.
So is there a connection between the two, mathematics and dance?
My goal is to inspire you to look at dance in a whole different light and to use your knowledge of maths next time you watch a dance performance.

DANCE FORMATIONS/ GEOMETRY

Groups of dancers use dance formations to position themselves on stage. Dancers in those formations use their bodies to form some geometrical shapes, but the formation itself forms a shape. So we stop looking at the dancers as individuals and start to look at the formation as a whole.
Our perception – the way we see things – makes us find a pattern in a structure made of individual objects. There is a term that deals with that property of our mind and its name is gestalt. Although there is a whole science behind gestalt, the main point is that our perception – the way we see things – gives us more than just a sum of individual objects and we start to see some patterns.
So, if we have two dancers on a stage it´s hard to say that they form a pattern, but if we have more of them dancing side-to-side we can say that they form: a line, a circle, some complex shape, or even a 3D shape.
Dancers are like geometrical points and by positioning themselves they form different shapes – the more points the better definition of the shape.

Geometry functions the same way. Two points on the plane can mean a lot of things like a line, a parabola, a circle, or a sine wave. But it´s when we have more points that it defines a shape or a function much better.
So the principle works for maths as well as for dance?
Formations often use the rule of symmetry.
Symmetry in mathematics is when a shape looks identical to its original shape after being flipped or turned.
The word symmetry is of Greek origin but in ancient Greece symmetry was referred to as a harmony of all measures. Isn´t that what dance strives to achieve? To accomplish the full harmony of movement through space, time and energy? – just as the ancient Greeks said – a symmetry!
We´ve seen the dance formations, but what about individual dancers?
Individual dancers often use symmetry. In fact, one could do hard without it.
Every pirouette could not be done without central symmetry. A pliè is a move that is supposed to be axially symmetrical.

FOCUS/ POINT OF INTEREST

Another term dancers and mathematicians use is focus/point of interest/accent. Or to describe it a bit: a spot that has all your attention at the given moment.
Why? – Because there is something interesting happening there.
The eye is drawn to the: interesting, unusual or dynamic and to the peaks and minimum and maximum values.
Of course, there is always something interesting happening on stage, but our attention can be drawn to the right or left, to the back or to the front, and even down or up.
Some dance formations use the position of the "dots/dancers" to emphasize certain points of interest.
So, what does math have to do with it?
Well, in mathematical statistics, the primary procedure is to observe certain events and "get a bigger picture" of those events.
By observing them, our attention is drawn towards interesting, unusual or dynamic things and to peaks and minimum and maximum values, because there is something interesting happening there.
In statistics, one has to focus on those "out of order" events in order to either document and explain events that already happened, which is called interpolation, or to predict events that could happen in the future, which is called extrapolation.
Interpolation and extrapolation are extensively used in the economy, banking, trade, but also in weather forecast and sports events.

DANCE CANONS/ VECTORS

So the attention of the observer is drawn towards a focus/accent. But what if the focus moves?
A point of interest can travel, skip and jump from one place to another and that gives us a different dimension of movement – dynamics in time.
There is a thing in dance which always gives us a special treat: a dance canon.

A dance canon is usually a repetition of the same dance fragment or phrase executed by several dancers who space it out in time.
It looks something like this:
Could you see the move? Your focus shifted from one dancer to the next and your perception noticed the move.
This move has a magnitude and a direction. – and so you get a vector.
Every vector is described by the magnitude – which, in this case, is the number of performers, and the direction – which is achieved by the perception of the moving point of interest.
The basic math operation with vectors is their addition.
The movement in this canon goes from the centre of the stage to the left and from the centre to the right.
The result is a front-to-back movement that resembles the resultant of both movements.
Vectors can also move in both ways – they can be in a positive or negative direction.
They also function perfectly well in three dimensions. The first dimension is shown by having dancers in the front do a right to left movement. The second and third dimensions are achieved by having two dancers spin above and behind the dancers in the front.

CONCLUSION

Mathematics and dance are connected in so many ways. They share the same language, the language of the universe, or as Galileo Galilei said, the language of numbers and emotions.
I can only hope that next time you see a dance performance, you will remember some of the things we discussed here and enjoy dance, the unique language of the soul, even more.

Materials sources and web sites used in this presentation:

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- uxknowledgebase.com
- wikipedia.org
- interaction-design.com
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- dama.math.hr
- The Concise Oxford Dictionary

MP82. THE REAL WORLD OF IMAGINARY NUMBERS

Barouchas Harry, Stratis Kounalakis, Moschonas Gerasimos, Xenikakis Nickolaos-Rafael, Papadopoulos Nikolaos, Evgenia Skagkou
Zanneio Experimental Lyceum of Piraeus, Greece

ABSTRACT

In this work we study the most famous numbers in mathematics that draw the interest even in recent years, namely the imaginary numbers. The set of latter numbers regards the difference between the sets of numbers which can be defined by the square root of a negative number and its complement set. What is the relationship between these two sets of numbers and what kind of mathematical problems were solved through it?
We present the history behind the imaginary numbers, explain their importance regarding various fields (from science to art and philosophy), describe their relationship with the real and complex numbers and explore the strength of mathematics (fractals) and how it can be applied to many important problems arising from different scientific areas. We then proceed by giving all the necessary definitions needed to the foundation of the set of imaginary and complex numbers and the operations between them, and we also present the definition and the graph of complex function of a complex variable. We also give the Euler's identity, the most famous identity of mathematical beauty, where the imaginary numbers are protagonists in. Finally, we conclude by studying why the imaginary numbers belong to the real world, solving crucial problems in it.

INTRODUCTION

At the current work we examine the set of numbers which have brought the attention for more than half a century. These numbers are the so called Imaginary numbers. Over the years, the Imaginary numbers influenced not only the Mathematical society but also other sciences and Arts have been attracted directly or not, due to their mysterious name.
In this paper we will focus on the work of the mathematicians as Gerolamo Cardano, Raphael Bombeli, Leonhard Euler, who initiated the research on the problem and were also involved in the study of these numbers. We will see at the same time how the study of Imaginary numbers resulted in the definition of complex numbers by William Rowan Hamilton. On the other hand, we will try to give answers to the following questions:

- How were the imaginary numbers created?
- Why are they so important?
- Are they related to Reality?

So, we will explore a tiny bit of the vast world of Mathematics.
A fundamental question to start with is: What is a number?

WHAT IS A NUMBER?

This question has its roots in the ancient years. Many philosophers such as Pythagoras, Plato, Aristotle, Kant and others were concerned about the notion of number trying to give a rigorous definition.
In mathematics, the notion of number has been extended over the centuries to include 0, negative numbers, rational numbers such as 1/2 and −2/3, real numbers such as $\sqrt{2}$ and π, and complex numbers which extend the real numbers with a square root of -1 (and its combinations with real numbers by addition and multiplication). So, the meaning of 'number', a concept first used to count physical things, to measure and label was extended by the introduction of negative numbers which were the reason for introducing the Imaginary numbers. Inevitably the question arose as to whether algebra would lead to another even more abstract type of number, for example, what is the square root of the square root of minus one?
A concept that, if you think about it too hard, will turn your brain upside down and twist it inside out. It is the solution to the equation:

$$x^2 = \sqrt{-1}, \text{ which is } x^2 = i.$$

Amazingly, the solution is $x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, a complex number.
But what is the meaning of Imaginary and complex numbers?

THE CREATION OF IMAGINARY NUMBERS – A BRIEF HISTORY

It is known that the mathematicians knew how to solve equations with a highest power of two from antiquity:

$$ax^2 + bx + c = 0.$$

When they came in contact with equations of the following form $x^2 + 1 = 0$, they thought that these kind of problems are unsolvable and they considered these equations impossible. The reason of course, was that the unknown number $\sqrt{-\mathbf{1}}$ appeared!!!!
Their major concern was the following: 'What sort of number, when multiplied by itself, is negative?' On one hand such a number cannot be positive since a positive number multiplied by itself is positive. On the other hand such a number cannot be negative as well since a negative number multiplied by itself is also positive.
Around 16th century, European mathematicians tried to solve equations with a highest power of three-cubics:

$$ax^3 + bx^2 + cx + d = 0$$

and then they came again face to face with the unknown square roots of negative numbers**!**

The Secret Method

In 16th century, the Mathematician **Del Ferro** (6 February 1465 – 5 November 1526, Bologna) was trying to find a method to solve the equation
$ax^3 + cx - d = 0$, using as an intuition the method of the quadratic equation
$ax^2 + bx + c = 0$, namely, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

So, he followed the same technical steps as quadratic equation and he found the solution:

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}+\sqrt[3]{\frac{d}{2}-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}$$

Del Ferro kept this formula secret until a little before he died, when he revealed it to his student **Antonio Maria del Fiore.**

Niccolo Fontana Tartaglia (1499/1500, Brescia – 13 December 1557, Venice) had successfully solved cubics but not the Del Ferro's form. However, he finally figured out how to solve these equations, but he kept the formula super secret!

Girolamo Cardano revealed the secret method

Girolamo Cardano (24 September 1501 – 21 September 1576, Pavia) heard about formula and pressured Tartaglia to share it, because he considered that there was no reason to hide it, so he decided to publish it in his book Ars Magna (1545).

However, along the way he came across a problem! To provide a better view of the situation that Cardano dealt, consider the following example: Let's apply the formula to the equation $x^3=15x+4$. Indeed, we have the equation $x^3-15x-4=0$, where a=1, c= -15 and d=4. Therefore, we have:

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}+\sqrt[3]{\frac{d}{2}-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}$$

$$x=\sqrt[3]{\frac{4}{2}+\sqrt{\frac{4^2}{4}-\frac{15^3}{27}}}+\sqrt[3]{\frac{4}{2}-\sqrt{\frac{4^2}{4}-\frac{15^3}{27}}}$$

$$x=\sqrt[3]{2+\sqrt{4-125}}+\sqrt[3]{2-\sqrt{4-125}} \qquad x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

Cardano noticed that a square root of negative number has appeared, something that led Cardano to stop Cardano his research.

Raphael Bombelli gave the solution to the problem

Cardano's student Rafael Bombelli (20 January 1526; died 1572, Bologna) gave the following solution to the square roots of negative numbers:

He introduced the new number $\sqrt{-1}$ and decided to treat the square roots of negative numbers just like positive and negative numbers, namely:

Algebra with x	***Algebra with*** $\sqrt{-1}$
$\sqrt{2\cdot 3}=\sqrt{2}\cdot\sqrt{3}$	$\sqrt{-25}=\sqrt{25}\cdot\sqrt{-1}=5\sqrt{-1}$
_2x+3x=5x_	$2\sqrt{-1}+3\sqrt{-1}=5\sqrt{-1}$
_5·x=5x_	$5\cdot\sqrt{-1}=5\sqrt{-1}$
_2+3x=2+3x_	$2+3\sqrt{-1}=2+3\sqrt{-1}$
$\sqrt{5}\cdot\sqrt{2}=\sqrt{10}$	$\sqrt{-5}\cdot\sqrt{-2}=\sqrt{-5\cdot -2}=\sqrt{10}$ ***(wrong)*** $\sqrt{-5}\cdot\sqrt{-2}=\sqrt{5}\cdot\sqrt{-1}\cdot\sqrt{5}\cdot\sqrt{-1}$ $=-1\sqrt{10}$ ***(right)***

He had, also, the following idea:

$\sqrt[3]{2+\sqrt{-\mathbf{121}}}$=a+b$\sqrt{-1}$ and $\sqrt[3]{2-\sqrt{-\mathbf{121}}}$=a-b$\sqrt{-1}$

therefore, we have:

$$2=a(a^2-3b^2) \qquad 11=b(3a^2-b^2)$$

and solving the system we find a=2 and b=-1.

Thus, a solution of the equation $x^3=15x+4$ is:

$$x=\sqrt[3]{2+\sqrt{-\mathbf{121}}}+\sqrt[3]{2-\sqrt{-\mathbf{121}}}=2-\sqrt{(-1)}+2+\sqrt{(-1)}=4.$$

Indeed, Bombelli has solved the Cardano's problem.
He (book I, Algebra, 1572) gave the name **'binomio'** to the number $a+\sqrt{-b}$, in close imitation of Euclid's theory of irrationals of the form a + √b in his Book X **(André Weil (Number Theory, 1984)).**
Bombelli commented for his work with the following words: 'It was a wild thought in the judgement of many' and 'The whole matter seemed to rest on sophistry rather than on truth' but the history of mathematics proved that Bombelli was not right.
Another problem, which was realized long after Bombelli's death was that the a positive number, such as 16, 9 , 4, is associated to the area of a square but can we say the same for the negative numbers?
But Mathematicians were not led down and they continued.

The word 'Imaginary' and the symbol 'i'

In 1637 **René Descartes** (31 March 1596 – 11 February 1650) described the square roots of negative numbers as **'imaginary'**.
A century later, **Leonhard Euler** (15 April 1707 – 18 September 1783) gave the number $\sqrt{-1}$ its own symbol, 'i' for 'imaginary'. He had wrote about the new numbers:
'All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.'
The time has passed, the concept of Imaginary numbers was established and the Mathematicians as well as the whole mathematical community were ready to give the definitions.

DEFINITIONS AND EXAMPLES

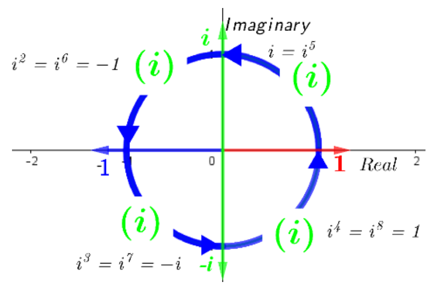
Imaginary numbers:

Let's take a look at some examples:
$\sqrt{-4}=2i$, since $\sqrt{-4}=\sqrt{4\text{x}(-1)}=\sqrt{4}\text{x}\sqrt{-1}=2\text{x}i=2i$
We can simplify this further, since, by definition,
$i^2=-1$,
and:

$i^3 = i \times i \times i = i^2 \times i = -1 \times i = -i$, $i^4 = i^2 \times i^2 = -1 \times -1 = 1$, $i^5 = i^4 \times i = 1 \times i = i$,
$i^6 = -1$ and so on.
In general, $\sqrt{-n} = (\sqrt{n}) \cdot i$: The square roots of negative numbers – which are all multiples of i – are collectively known as the 'imaginary numbers'.

The visualization of imaginary numbers

If we start with the number 1 and multiply by i, algebraically we get i,
which geometrically corresponds to
a 90° rotation from 1 to i.
Multiplying by i again results in i squared, which, by definition is minus 1.
So, the insight here is that imaginary numbers do not exist apart from the real numbers.



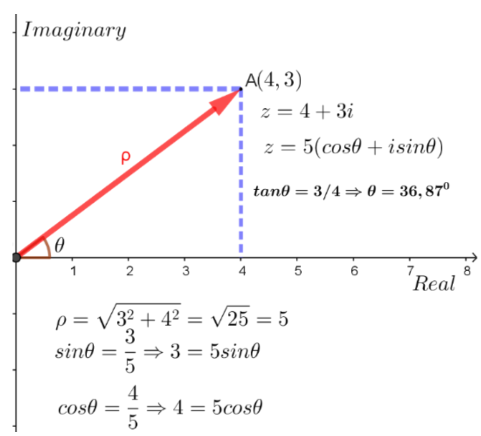
They are the natural extension of our number system from 1 dimension to 2. Numbers are two dimensional.

Complex numbers

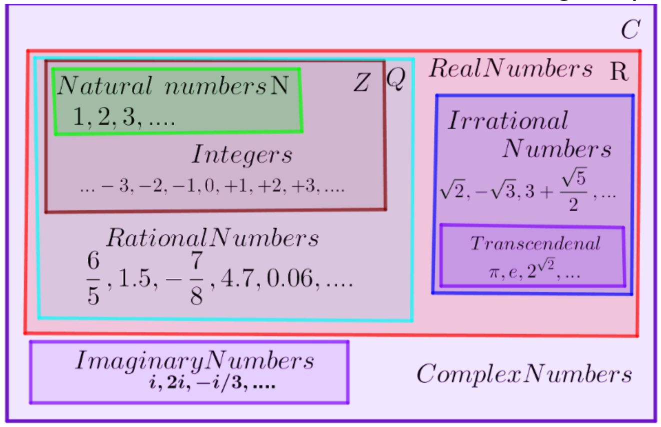
When a real number is added to an imaginary number, the hybrid form, such as 3 + 2i, is called a 'complex number'.
All complex numbers are of the form a + bi, where a and b are real numbers and i is $\sqrt{-1}$.
Since you can't add a real number to an imaginary number in the traditional sense, the plus sign is just a way of separating the two parts.
A complex number is considered a single number with two parts, its real and imaginary ones. If the real part is zero, the number is purely imaginary, and if the imaginary part is zero, the number is purely real.

The visualization of comlex numbers

In general, a complex number can be visually represented as a pair of numbers (a, b) forming a vector on a diagram called an Argand diagram, representing the complex plane. Together, ρ and θ give another way of representing complex numbers, the polar form, as the combination of modulus (ρ) and argument (θ) fully specify the position of a point on the plane. Recovering the original rectangular co-ordinates from the polar form is done by the formula called trigonometric form z=ρ(cosθ+isinθ) **(John Wallis 1616-1703).**



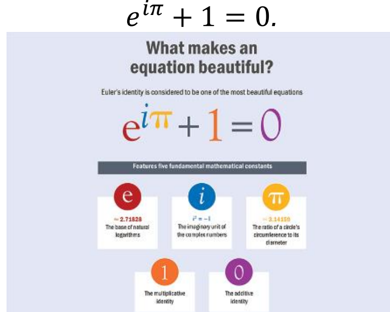
Finally, we have the creation of a new set of numbers, as the diagram points:



THE IMPORTANCE OF IMAGINARY NUMBERS

- **Mathematical Beauty:** Imaginary numbers are protagonists of two of the most famous examples of mathematical beauty. One is a picture (of which more later) and one is an equation, known as Euler's identity, which in 2003 was sprayed on the side of an SUV in an eco-terrorist attack on a Los Angeles car dealership. The nature of the graffiti led to the arrest of a physics PhD student at Caltech. 'Everyone should know Euler's [identity],' he explained to the judge. He was correct, but one should nevertheless refrain from daubing it on cars.
Euler's identity is the 'To be or not to be' of mathematics, the most famous line in the oeuvre and a piece of cultural heritage that resonates beyond its field:

$$e^{i\pi} + 1 = 0.$$



But why this identity is considered the most beautiful?
Because each number emerged from a different area of enquiry, yet they unite with perfection. Indeed,
1 is the first counting number, 0 is the abstraction of nothing, e is the exponential constant, i is the square root of minus one, π is the ratio of a circle's circumference to its diameter.
You couldn't have predicted a more immaculate synthesis of mathematical thought. Beauty in mathematics is about elegance of expression and making unexpected connections. No other equation is as concise or as deep.

- **The Fundamental theorem of Algebra:** Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

The first proof of the theorem was given by Carl Friedrich Gauss (1777-1855) in his Ph.D. Thesis (1799).

- **Sir William Rowan Hamilton (1805-65)** in an 1831 memoir defined

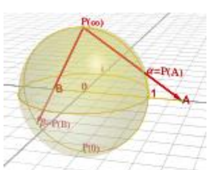
ordered pairs of real numbers (a, b) to be a couple. He defined addition and multiplication of couples:

$$(a, b)+(c, d) = (a + c, b + d) \text{ and } (a, b)(c, d) = (ac - bd, bc + ad).$$

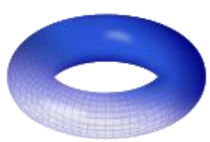
This is in fact an algebraic definition of complex numbers. For many years Hamilton sought to construct a theory of triplets, analogous to the couplets of complex numbers, that would be applicable to the study of three-dimensional geometry. Then, on October 16, 1843, Hamilton suddenly realized that the solution lay not in triplets but in quadruplets, which could produce a noncommutative four-dimensional algebra, the algebra of quaternions.
As J. Hadamard has put it:
'The shortest path between two truths in the real domain passes through the complex domain'.

- **Riemann surfaces:** Riemann surface for the function f(z) = √z. The

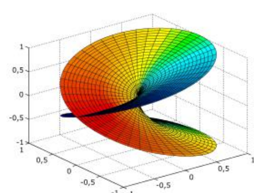
two horizontal axes represent the real and imaginary parts of z, while the vertical axis represents the real part of √z. The imaginary part of √z is represented by the coloration of the points. For this function, it is also the height after rotating the plot 180° around the vertical axis.



Riemann sphere



Torus



Riemann surface

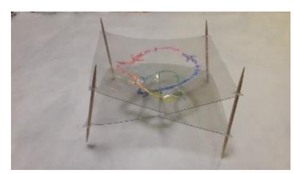
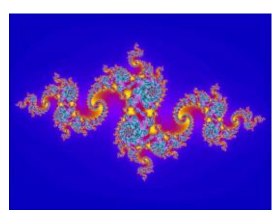


Photo of our construction of a Riemann surface

THE REAL WORLD OF IMAGINARY NUMBERS - APPLICATIONS

In Mathematics

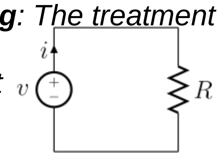
- ***Fractal geometry: The Maldelbrot set*** _is a_ popular example of a fractal formed on the complex plane. It is defined by plotting every location c where iterating the sequence $f_c(z) = z^2 + c$ does not diverge when iterated infinitely. Similarly, Julia sets have the same rules, except where c remains constant.



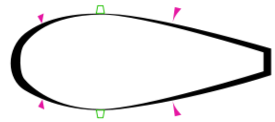
- ***Algebraic number theory:*** _any nonconstant polynomial equation (in_ *complex coefficients) has a solution in C. A fortiori, the same is true if the equation has rational coefficients. The roots of such equations are called algebraic numbers.*

In Physics

- ***Electromagnetism and electrical engineering**: The treatment of resistors, capacitors, and inductors can be unified by introducing imaginary, frequency-dependent resistances for the latter two and combining all three in a single complex number called the impedance.*



- ***Fluid dynamics:** In fluid dynamics, complex functions are used to describe potential flow in two dimensions.*



- ***Quantum mechanics**: The complex number field is intrinsic to the mathematical formulation of quantum mechanics.*

In Art

- ***Fractal art**: is a form of algorithmic art created by calculating fractal objects and representing the calculation results as still images, animations, and media.*

- ***Lewis Carroll (Charles Dodgson) in Alice in wonderland**: In chapter 'Mad Tea Party' lampoons Hamilton's quaternions, its title a pun on 'mad t-party', where t is the scientific abbreviation for time. At the party, the Hatter, the March Hare and the Dormouse are rotating around the table just like the imaginary numbers i, j and k in a quaternion. The fourth guest, Time, is absent, so there is no time for washing up. When the March Hare tells Alice that she should say what she means, she replies that 'at least I mean what I say – that's the same thing, you know'. Yet word order does change the meaning, just as the order of i and j in quaternion multiplication changes the result.*

Philosophy of Complex numbers

- *If you wonder about the ontological undecidability of non-existing objects in the real of the "possibility" in contrast to "actuality", then you will have to consider the Kantian view of phenomenal objects as the result of interaction between external and internal or to postulate the metaphysics for non-existent objects as a complete equality between subject and object. Everything clear?*

- *The above statement illustrates a difference between Mathematicians and Philosophers: while Mathematicians only need paper, pen and a waste basket to work, Philosophers can even do with paper and pen alone ....*

CONCLUSION

So, that's why imaginary numbers have been a great breakthrough for the field of mathematics and science!!!!

Thus, Bombelli was unfair when he said that 'the whole matter seemed to rest on sophistry rather than on truth'!!!

AKNOWLEDGMENTS

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MP84. UNLIMITED CAPABILITIES: A REPORT ON VARIOUS SUBTYPES OF ARTIFICIAL INTELLIGENCE

Philippos Ganos, Antonios Panagakos, Alexios Papadopoulos, Christos Petropoulos
The American College of Greece-Pierce, Greece

ABSTRACT

The purpose of this research is to dive into certain subtypes of Artificial Intelligence. Namely, the use of AI in space exploration, paradoxes that exist in AI, while also showing how it can affect humanism in the future. Regarding space exploration, we are researching how AI can be used to collect data from planets we have visited, as well as how it will help us explore even more parts of our solar system in the future. Another subtype that is addressed is Deep Learning and Large Scale Machine Learning, which will be playing a pivotal role in every computer in the future. Lastly, we take a look at how AI is used in the video gaming industry and how, even in the most complex of games, it can outmatch any human opponent. In particular we will explore some of the most common gaming AI learning methods such as decision trees and finite machine learning.

Artificial intelligence is the theory an development of a computer to perform tasks that would normally require human intelligence, such as visual perception, speech recognition, decision making, and translation between languages. Artificial intelligence has revolutionized the world today with self-driving cars or small robots. But where did it all start...at the start A.I. was doing simple human tasks as calculations and as the time passed by and through its development it has now achieved huge breakthroughs in the medical department; and it doesn't stop here, A.I. is growing in a very fast speed. The evolution of A.I. more analytically:

I. 1955 AI IS BORN • Term "A.I." is born by John McCarthy who threw light on the question CAN A MACHINE THINK?
II. 1961 UNIMATE • 1st industrial robot, Umimate replaces humans on the assembly line
III. 1964 ELIZA • Pioneer chatbot developed by Joseph Weizenbaum at MIT holds conversations with humans
IV. 1977 DEEP BLUE • Deep Blue, a chessplaying computer from IBM defeats world chess champion Garry Kasparov
V. 1998 KISMET • KISmet is an emotionally intelligent robot that detects and responds to people's feelings (C. Breazeal / MIT)
VI. 1999 AiBO • AiBO is the first consumer robot pet dog with skills and personality that develop over time (from Sony)
VII. 2002 ROOMBA • The first mass produced autonomous robotic vacuum cleaner that learns to navigate and clean homes (from iRobot)
VIII. 2011 SIRI • Apple integrates Siri, an intelligent virtual assistant with a voice interface, into the iPhone 4S

IX. 2016 AlphaGo • Google's AI AlphaGo beats world champion KeJie in the complex board game of Go, notable for tis cast number of possible positions
X. 2017 UBER • UBER pilots self driving car program in Pittsburg

Artificial intelegence in machine learning

"I know that I know nothing," said Socrates. A quote so bizarre, yet so deep in meaning. Two thousand years later, this is the mindset of Machine Learning. Machine Learning, usually referred to as "data mining", is the study of algorithms that computers use in order to take out a task without having specific instructions to do so. Much like humans, computers that use Machine Learning learn through their past experiences. The ways they learn are called "Learning Algorithms". There are three types of Learning Algorithms.

1. Supervised learning
2. Unsupervised learning
3. Reinforcement learning

Supervised Learning algorithms create a model data set which consists of labeled data associated with certain features. Every time the computer system has to recognize the type of the new data input, it checks it's pre-existing labeled data in order to match some of its traits with its preexisting ones. During this procedure, the computer is trained each time new labeled data enters the data set. In order to ensure that the algorithm functions properly, a human is assigned to moderate and supervise -hence the name- the algorithm.

In order to understand Supervised Learning algorithms better, visualize the following scenario: you have three items; a pen which weighs 8 grams, a pencil which weighs 11 grams, and a marker which weighs 18 grams. If you were to enter data regarding the weight of each item in a Supervised Learning algorithm, the algorithm would check its labeled data set to match the weight with the correct feature -in our case, the item. Presumably, it matched a weight of 8 grams with a pen, a weight of 11 grams with a pencil, but it could not find any data to match 18 grams with. In that case, the algorithm would train itself and learn that 18 grams correspond to a marker, referring to that newly created data from now on.

On the other hand, Unsupervised Learning algorithms vastly differ from Supervised Learning. There is no human supervisor, there is no training data set, and the algorithm is dependant on a set of *unlabeled* data. These algorithms sole goal is to interpret input data and match it with unlabeled data in order for the desired output to be achieved. For the computer to recognize the input data, it compares it to their pre-given sets known as "clusters". The algorithm identifies the pattern that is prevalent in the data it is fed and then, essentially, it "graphs" its answer and outputs the desired outcome.

The last of machine learning is reinforcement learning. It shares some traits with both Supervised and Unsupervised algorithms -mostly with the latter-, but it vastly differs. In reinforcement learning algorithms, the computer only learns through its mistakes, much like humans. Such algorithms do not require pre-established data sets, except for some necessary data in order for the algorithm to function properly. Whenever a computer that uses reinforcement learning data is misinterpreted and the algorithm is strengthened. In the end, reinforcement learning algorithms are dependant on their own sets.

A neural network is an algorithm that is able to receive multiple input data and processes to one or multiple outputs; it consists of small units called neurons. Neurons are grouped into layers in order to process and calculate many different outcomes that might be plausible. Each neuron is connected to every single neuron from the next layer. These connections are called "weighted neuron connections" and are calculated using an activation function, which differs based on the neural network's structure. The only way for data to be transmitted through a neural network and the input is stored as a corresponding numerical value, for it to be calculated, the neuron's bias value is needed. The bias value of a neuron is calculated by the sum of all previous neurons, which are multiplied with their connection's weight. One of the simplest and most common activation functions is:

$$\frac{1}{1 + e^{-x}}$$

Artificial intelligence in gaming

Gaming A.I. is as old as artificial intelligence itself and it is commonly used to determine the behavior of non-player character or in other words NPCs. The gaming industry keeps evolving day by day so does A.I. and its learning methods keep improving. Some of the oldest and most used learning methods used in gaming A.I. is decision trees and finite machine learning.

In simple words, decision tree learning is a type of supervised algorithm one of the most widely used and practical methods for reaching a general conclusion from specific examples. Moreover, decision trees often mimic the human level thinking so it's so simple to understand the data and make some good interpretations. With this learning method, we attempt to map attributes of data observation to the specific target.

Decision tree learning has three main components: the former inputs which are known as nodes and could be a question or a specific condition (e.x. Is the target visible? are the target's hit points low?). After each node, there is the branch of the tree called edges which are the values of the nodes and are usually represented as Yes/No options. Lastly, there is the latter part of the tree or we can say the leaves of the tree which is the former output and the end action of the branch that A.I. followed. The main goal is to create a tree for the entire data and process a single outcome. (ex. diagram 1)

So, the question is how do we build a decision tree through code language? There are a couple of algorithms, but we are going to concentrate on a few such as CART (Classification and Regression Trees) that can be used for predictive modeling problems.

Each node represents a single input variable (x) and then the leaf of the node contains a variable (y) in order to make a prediction. Given data with two inputs for example height in meters and weight in kilograms (as we see on diagram 2) the algorithm can predict the output which in this case is the sex of the person as a male or a female. This is the final form of the decision tree:

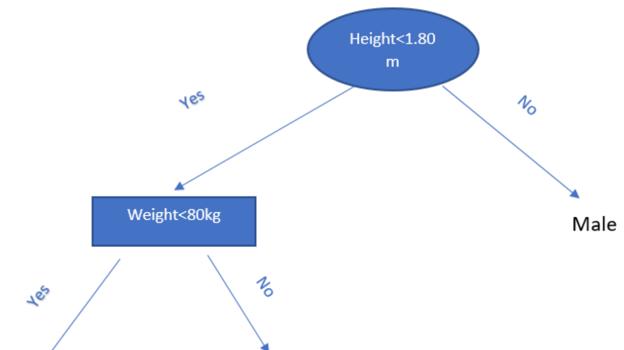

Diagram 3

A finite-state machine or for short FSM is a type of A.I. learning method based on a hypothetical computation system that consists of one or more states. However only a single state can be active at the same time so the machine should go from one state to another in order to perform an action, it is commonly used in videogames to represent an enemy's NPC (Non player character) way of thinking and acting or even a video game can be represented as a FSM by means of a graph, in which the nodes are the states and the edges are the transitions. Each edge has a circumstance when the action should happen which if it is true the machine will change from the current state to another. All these factors are used to create very complex algorithms in order for A.I. to work properly. Now an easier to understand approach is depicted in the below graph:

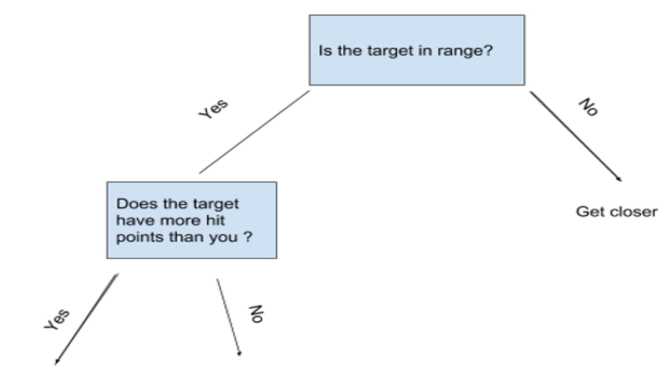

Diagram 1

Video game are state machines too, everything that appears on our screens are states that describe what the game is thinking about. The are described at programming so they can be hopelessly complicated. But we can find out a lot about the game's state just by looking on the screen though. Let's analyze this example from Super Mario Bros for instance.
Here is what the game state consists of:

Mario's position: (50, 200)
Mario's size: Small
Stage number: 1-1
Camera angle: 0 pixels
Score: 0 points
Time: 400 seconds
Coins collected: 0
Enemies on screen: none
Blocks on screen: none
Coins on screen: none

Not all elements that are in the game state appear on the screen, but the state describes every bit of data that the game needs to know on any given time for example, Mario has 3 lives, but it is not on the screen (Lives: 3).

Plenty of actions can change this state usually with player's input, for instance if we press the key W (action) Mario will jump and its position will change (new state). And also because the game takes place in real time actually the passage of time itself is an action that changes the game's state too. Or even if an enemy shows up it will change the game's state and it has to be described with detail. (see below).

Mario's position: *(60,250)*
Mario's size: Big
Stage number: 1-1
Camera angle: 0 pixels
Score: 1800 points
Time: *308 seconds*
Coins collected: 3
Enemies on screen:
Enemy 1:(120,190)
Enemy 2:(140,190)
Blocks on screen: none
Coins on screen: none
Lives: 3

The A.I. Paradox

With the A.I. developing faster than ever it's not beyond imagination for the society to be automated. The moment we are speaking the A.I. is able to drive, understand speech and all lot of other things in a sorter amount of time. The time is not so far away that the A.I. will be smarter than people. The A.I. will be transformed to a super intelligent machine and it is called "superintelligence". This is going to happen when the A.I. will be able to feel and act like humans.

This is going to make humans completely useless. At this point that the A.I. will have already reached superintelligence things won't be looking good for humans as they would be at this point almost stupid against the intelligence of the A.I. We should be scared about the A.I. of eliminating either on purpose or by mistake. The paradox is that the A.I. may sometime in the future achieve this superintelligence and become smarter than its inventor, the humans.

Even though an A.I. can do a lot of things it cannot do 1 thing: According to Patrick Winston, a professor of AI and computer science at MIT he says that the A.I. doesn't have the ability to understand the physical world well enough to make predictions about basic aspects of it which means that the A.I. doesn't have common sense. The paradox is that the A.I. needs the common sense to function.

According to a principal named Hans Moravec the A.I. isn't still here. That's because of a paradox called the Moravec paradox: -we can teach machines to solve the hard problems but it's the easy ones that are the difficult. For example they might be able to beat a champion at chess, but they can't pick the right toy off the shelf.

The Polanyi's paradox is summarized in the sentence: "we can know more than we can tell" i.e. many of the tasks we perform rely on tacit, intuitive knowledge that is difficult to codify and automate. For example, we recognize faces without knowing how we do it • Example: If algorithms of chair identification are implemented in a machine, it will identify a chair based on chair properties ie as legs, arms, seat, back etc. But, not all chairs possess the same properties - no legs (beanbags) no back (stool) and so on. In this case, the machine would fail to recognize them as chairs whereas humans can tell that through knowledge - which cannot be ascertained as to why it happened that way.

Another question that remains is can A.I. be killed by A.I. Theoretically if you force the A.I. into an infinite loop this would essentially freeze the computer's consciousness and "kill" it

A.I and space

Experts believe that space exploration will be crucial in the future. Artificial Intelligence and its development will help humanity become acquainted with even more about our solar system and other planets, apart from all the things we already know. So, A.I will make our efforts much easier. The scientists' goal is to create robots, that won't be controlled by anyone and which they will help them, collaborating with them.

However, in order to explore space, we need to work on artificial intelligence and develop it. First of all, by using A.I, we can have some special skills, whom we don't actually have without it. Such skills are faster accomplishment of our goals and reduce of the cost needed for the operation of a specific work. Apart from these, it prevents men from working into dangerous places, so it improves workers' safety. Story Musgrave, who was a NASA astronaut, says that almost seventy percent of the whole crew gets ill during their transmission into the space. He also says that fifty percent of them are so ill, that they throw up, because they haven't been familiar with so demanding conditions. Last, but not least, its decision-making helps us make productivity better, according to the worldwide known Professor Gao. As we all easily understand, this is crucial for the success of the scientists' efforts. Moreover, the fact that robots will be self-piloted is very important for another reason; at a specific point in space, communication between the space station and the people in the rocket is impossible and they will then have to do their work (explore , collect information) alone , without any assistance from the ones who are on Earth. So, if robots and, more generally, machines, based on artificial intelligence, are made, the experts' job will get easier.

Furthermore, if A.I participates in space exploration, we will have the chance to complete missions that have not been finished yet. For example, both the Spirit and Opportunity rovers, which were launched in 2003, had an A.I system that drove them and allowed them to explore the surface of Mars. That system was called Autonav. Moreover, even if a lot of people may not know it, the A.I has developed a lot since it was first used for space exploration. For instance, one of the first things to use artificial intelligence, in order to help us learn more about space, was named "Earth Observer 1 (EO1) satellite". It contributed to the analysis and response to natural phenomena (earthquakes, floods, volcanic eruptions), even before the ground crew noticed them or was aware of their existence. It was also capable of capturing photos and collecting information about these natural occurrences. As a result, the fact that by the early 2000's, humanity had made such a functional and useful self-piloted satellite gives us hope and makes us expect impressive accomplishments in the future.

When it comes to the places we have already sent probes to, artificial intelligence will help us collect and analyze the data. This will offer us the chance to explore them further, even if it is in a theoretical level. Almost every single specialist believes that without the A.I system's help, we will never complete space exploration. That means that we are totally depended on the improvement of A.I. Additionally, because NASA is planning trips to space that will take a long, we should have robots, to replace humans, if something goes wrong. Overall, this is another crucial reason for which we should try our best, so as to develop artificial intelligence as much as we can. According to an engineer, who works for NASA, if people don't finish their work soon, and try to send a probe without any autonomy at all, it will be very likely for it to face a really serious difficulty or even get stuck in ice. That's why there is a high need of advanced technology; otherwise, our efforts will be truly complex, if not impossible. The use for A.I for such reasons is cheaper. Moreover, as robots may be portrayed in a wealth of comics and science fiction movies, they have a lot of functions. In the future, space exploration will be based on artificial intelligence, which impacts in the research, the spotting and information's analysis.

Nowadays, we have sent a system, that is self-piloted, on Mars. It is called AEGIS. It can select what is useful to investigate, and then targets the cameras there; all by itself and with the use of artificial intelligence. However, it is believed that when the next generation of A.I will be used, it will have the potential to move vehicles, like spacecrafts, aid the scientists regarding their studies and, more generally, complete scientific tasks, which are difficult for people.

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MP86. EUCLIDIAN VS NON EUCLIDIAN GEOMETRY: IS GEOMETRY ONE SIDED?

Anastasia Efraimoglou, Eleni Fouka, Evridiki Koletsou, Victoria Martin Veneti, Eirini Sigalou, Nikitas Stathopoulos
The American College of Greece-Pierce, Greece

ABSTRACT

Geometry is represented all around us in nature, in many different forms. From Euclidian to any non-Euclidian geometry, we live in a world consisted of geometric shapes that follow geometric rules. In this paper some areas of geometry are being examined, namely Euclidian Geometry, Differential Geometry, Algebraic Geometry, Hyperbolic Geometry and Elliptic Geometry. The definition, the history and the basic axioms of each area of geometry are presented and the usefulness of each geometrical system is examined. In each case we present some models as well as applications and examples. It is hard if not impossible to imagine a world without geometry and that therefore proves its importance. The above is also the reason for which we chose to examine geometry in our project.

Euclidean Geometry

Euclidean geometry is the study of shapes on flat surfaces. It's based on 5 basic axioms, 23 definitions and 7 or 9 common concepts and through them 465 propositions are proved. The definitions are essentially the definitions of concepts such as point, straight line, angle, right angle, circle, etc. Some of them (such as the circle, parallel lines, etc.) are accurate and useful, while others (like point, line, etc.) are unclear and of no particular use. The axioms refer to geometric propositions or properties that associate unspecified terms and whose truth is accepted without proof. These are:

1. Given any two points, you can draw a straight line between them.
2. Any line segment can be made as long as you like.
3. Given a point and a line segment starting at the point, you can draw a circle centered on the given point with the given line segment as its radius.
4. All right angles are equal to each other.
5. The angles of a triangle always add to 180 degrees.

The fifth axiom can be written in lots of different ways and it is also known as parallel postulate. This axiom is what separates Euclidean geometry from the others because other geometries assume that is false and replace it with another form. Common concepts are propositions for which are true (like axioms), that don't refer to geometric qualities or relationships, but are mainly propositions of logic. Proposals are statements whose validity may be proved, following a logical process, from axioms and common concepts, and from other already proven proposals.

In the "Elements" there are three evidence methods; the synthetic where in order to prove something, we use the definitions, the axioms and other already known propositions and with logical reasoning we end up in the proof; th method of Reduction ad Absurdum where we begin with a hypothesis that is contrary to what we want to prove. Using the axioms and other known propositions, we come up with a proposal that is in contradiction with what we already know. The contradiction arises because we started from a mistaken hypothesis, and it is withdrawn if we accept that the proposition we wanted to prove is true; the analytical method which is based on the following methodology: we accept that the proposition P, to be proved, is correct. Then, we infer that other proposition or propositions are correct. Since the latter are correct, then, obviously, the first proposition is correct too, since we are able, by synthesis, to be led to that first one.

Geometry's usefulness is everywhere; angles, shapes, lines, line segments, curves and other aspects of geometry are in every single place you look. As a result Geometry has applications in many aspects of real life. To begin with, Geometry is included in every section of science and even in other areas. For instance, computer imaging, something that is used these days for designing, creating video games, animations and lot of other things, makes use of geometric concepts. In addition, architects and builders use Geometry to calculate areas before they start making plans for buildings as it helps them decide what materials to use and what design principles to follow. In addition, this plays a vital role later, in the construction process itself. In Greek Geometry is translated as “earth measuring”. By that we can understand that Geometry has to do with everything that is on earth and not only.

The reason why we study Euclidean geometry is because we can use it to solve all kind of real-world problems. It’s also more easily understood by the average person and it’s the type of geometry on which all the others have been based. Furthermore, it helps us understand the needs of ancient Greeks and the progressiveness of the historical period.

Elliptic Geometry

Elliptic or else Riemannian geometry is a non-Euclidian Geometry that satisfies all Euclid's postulates except the 5^{th}; therefore the parallel postulate does not hold, which states that "through any point in the plane, there exist no lines parallel to a given line".Felix Klein was the first who saw clearly how to rid spherical geometry of its one blemish; the fact that 2 lines have 2 common points instead of one. And Bernhard Riemann, in 1854, was the one who eventually formed this type of geometry at its final state. In elliptic geometry the plane has a positive curvature (curvature is any of a number of loosely related concepts in different areas of geometry) and can be studied in two, three, or more dimensions.Still, as explained by H. S. M. Coxeter; The name "elliptic" is possibly misleading. It does not imply any direct connection with the curve called an ellipse, but only a rather far-fetched analogy.

In order to achieve a consistent system in elliptic geometry (in mathematics and in particular in algebra, a linear or nonlinear system of equations is consistent if there is at least one set of values for the unknowns that satisfies every equation in the system). However, the basic axioms of neutral geometry (a.k.a. absolute geometry) Geometry without parallel axiom.) must be partially modified. Most notably, the axioms of betweenness (p.e. if a*b*c are three distinct points lying on the same line then one and only one of the points is between the other two) are no longer sufficient (essentially because betweenness on a great circle makes no sense), and so must be replaced with the axioms of subsets (=For every set and every condition, there corresponds a set whose elements are exactly the same as those elements of the original set for which the condition is true). Also, the parallel postulate does not hold, therefore: Given an arbitrary (by chance-random) infinite line *l* and any point *P* not on *l*, there does not exist a line which passes through *P* and is parallel to *l*.

Elliptic geometry can be visualized as the surface of a sphere on which "lines" are taken as great circles. Also, there is a variety of properties that differ from those of classical Euclidean plane geometry. For example, the sum of the interior angles of any triangle is always greater (>) than 180°. For example, all lines have the same finite length π, the area of the elliptic plane (A flat surface that is infinitely large and with zero thickness) is 2π, the sum of the angles of a triangle is always (greater than) > π. Therefore, this geometry satisfies all Euclid's postulates except the 5th.

Models of Elliptic Space (elliptic space: a space endowed with a non-Euclidian Elliptic Geometry); Spherical geometry (Spherical geometry replaces the standard flat plan with the plane being the surface of a sphere) gives us perhaps the simplest model of elliptic geometry. Points are represented by points on the sphere. Lines are represented by circles through the points.

Some of the Applications of Elliptical Geometry; *Navigation*: Spherical geometry is used by pilots and ship captains as they navigate around the globe. *Spherical Astronomy* (– Positional Astronomy and Space Exploration): the application of spherical trigonometry, and aims at determining stellar positions on the celestial sphere. *Spherical Geometry*: p.e. It has been proven that the shortest flying distance from Florida to the Philippine Islands is a path across Alaska – even though the Philippines are at a more southerly latitude than Florida! The explanation is that Florida, Alaska and the Philippines lie on the same great circle and so are collinear in spherical geometry.

Differential geometry

The central mathematical problem with which mathematicians were concerned was the exclusion of a mathematical explanation with the precise distinction of a curved line from the line and the surface from the level. The question that required an answer was what exactly does the curvature of a curve or surface mean and how can it be measured? In the second half of the 17th century with the discovery of differential and total calculus by Newton and Liebnitz, the problem began to be solved. Differential calculus owes its existence to the need to measure changes in the length of the vector of a curve. The use of Differential Calculus in the study of Geometry led in the creation of a new branch, Differential Geometry.

Differential Geometry is a branch of mathematics that uses differential calculus to study the geometric properties of curves and surfaces. Differential geometry consists of the following branches; Riemannian geometry, Pseudo-Riemannian geometry, Finsler geometry, Symplectic geometry, Contact geometry, Complex, Khäler geometry, CR (Cauchy - Riemann) geometry, Differential topology and Lie groups. Differential geometry developed much more rapidly during the 18th and 19th centuries. In the development of Differential Geometry, the German mathematician Carl Friedrich Gauss had fundamental complicity. Also, major contributions in Differential Geometry’s development had I. Newton, G. W. Leibniz, L. Euler, G. Monge, J. Bernoulli, A. C. Clairaut, F. Frenet, J. A. Serret, Ch. Dupin, J. Bertrand.

Gauss with two of the most important theorems of classical geometry contributed as best as he could to surface theories but also laid the foundation for the future research development of the branch. With the first theorem, Theorema Egregium, Gauss proved that there is a kind of a curve that can only be measured by an observer on the surface and that can vary from surface to surface. Although for the case of curves, an observer on a curve cannot recognize whether it is in a straight line or a curve, it does not happen the same for the case of the surfaces. Namely, we can understand that earth is spherical without traveling in space.

The second theorem is named Gauss-Bonnet as it was completed by Bonnet, a student of Gauss. The theorem states that the study of local properties of a surface is not sufficient for the understanding of surface-related phenomena at a global level. Bernhard Riemann was a German mathematician that used the basic ideas of Gauss in surface theory to extend them into manifolds which are the language of differential geometry with applications in several other branches of mathematics. In addition, Riemann has shown that non-circumferential geometries can be considered as special geometry of surfaces, which are not necessarily implanted in the Euclidean space.

The study of curves, besides its mathematical interest, has important applications in physics such as in Einstein's theory of relativity since differential geometry is the language that expresses his theory. In addition, differential forms are used in the study of electromagnetism, in Lagrange's and Hamilton's engineering. Riemannian geometry and contact geometry have been used for the construction of the formalism of geometrothermodynamics. Also differential geometry is applied in chemistry, biophysics, economics, statistics, engineering, navigation, satellite systems, wireless communication, computer graphics and in image processing.

An example of an application of differential geometry in our everyday life is what happens when we try to wrap a ball as a birthday present. One cannot avoid the creasing of the paper. This is due to the paper having zero curvature, and the ball having positive curvature, as we can derive from the Theorema Egregium.

Algebraic geometry

To give a simple definition it is a branch of mathematics based on analytic geometry that uses geometry in order to solve algebraic problems and vice versa
As for the applications of it we can find them in statistics and economics generally, robotics, engineering, error-correcting codes.
But why do mathematicians study it?

1. It is highly challenging.
2. Geometers study algebraic geometry so that they can do algebra. And algebraists study algebraic geometry so that they can do geometry.
3. A bridge between seemingly different disciplines: from geometry and topology to complex analysis and number theory.
4. It attempts to give a quantum counterpart to classical geometries.
5. In order to solve several physic theories.

To give an example to make it more clear. In order to create an algorithm for a path you are going to walk. If there are 2 paths (the first path stops where the second one starts then the product of the 2 paths is defined by: the product of the two paths is their union. This operation is equal to the equivalence relation and induces an corresponding product to the total. With that product the total is a group and it is called fundamental group of the x.

Geometrical place – Togliatti surface

As for the history of algebraic geometry, its roots date back to Hellenistic Greeks (5th century bc). Because of the Delian problem, to construct a length x so that the cube of side x contains the same volume as the rectangular box a^2b for given sides a and b. Menaechmus started thinking that the problem is geometrical and can be solved by intersecting the pair of plane conics $ay = x^2$ and $xy = ab$. In mathematics, a plane is a flat, two-dimensional surface that extends infinitely far. Archimedes and Apollonius thought of the problem different than Menaechmus, they studied the problem based on conic sections (in mathematics, a conic section, or simply conic, is a curve obtained as the intersection of the surface of a cone with a plane) and also used coordinates.

The actual start of algebraic geometry came from Arab mathematicians, that they were able to solve geometrical problems with pure algebra and then to attach the result geometrically (e.g. Cubic equations). Later on Omar Khayyám (born 1048 A.D.), a Persian mathematician, and Iranian mathematician and astronomer Sharaf al-Din al-Tusi's wrote "Treatise on equations" and this has been described as the official start of the algebraic geometry. So after that time algebraic geometry continued to grow in the times of Renaissance, at the 19th and the 20th century until now. There was developed a foundation for algebraic geometry based on contemporary commutative algebra (studies commutative rings their ideas and modules over such rings)

In the 1950s and 1960s the use of the sheaf theory started (a tool for systematically tracking locally defined data attached to the open sets of a topological space). Later the idea of schemes was really worked out. The next decade there was a rapid development in algebraic geometry. Lastly Bruno Buchberger introduced the theory of Gröbner bases (a kind of generating set of an ideal in a polynomial ring K[x1, ..,xn] over a field K.)

Hyperbolic Geometry

Hyperbolic geometry, also called Lobachevskian Geometry, is a non-Euclidean geometry which was created in the 19^{th} century and is based on surfaces with negative curvature. It is called hyperbolic geometry because of one of its very natural analytic models. Hyperbolic geometry emerged from the systematic study of Euclidean geometry and the attempts to find a proof for the 5th postulate, also called the parallel postulate. As mathematicians were studying, they came out with different conclusions from the ones that were already established in Euclidean geometry. Hungarian Janos W. Bolyai (1802 – 1860) and Russian Nikolai I. Lobachevsky (1793 – 1856) were the first ones who discovered these different conclusions and the ones that were able to prove them as well. Hyperbolic geometry emerges when we replace the 5th postulate with the hyperbolic parallel postulate. Lobachevsky and Bolyai also had different opinions about the rest of the Euclidean axioms. Firstly, the sum of the measures of the angles of a triangle is less than 180°. Also, similar triangles are congruent triangles. The postulate that replaces the discarded parallel postulate of Euclidean geometry states that through a given point not on a line there are infinitely many lines parallel to the given line. Another axiom is that each angle of a Saccheri quadrilateral measures less than 90°. A Saccheri quadrilateral is a quadrilateral with two equal sides who are perpendicular to its base. Lastly hyperbolic geometry claims that no quadrilateral is a rectangle. The first model of hyperbolic geometry is the Pseudosphere. It was suggested by E. Beltrami in 1868 and it is the surface that is produced by the rotation of a tractrix circa its asymptote and is composed of one or two funnels. In the case of this model we regard as points of the hyperbolic plane the points of the surface of the Pseudoshere, while as lines we regard the tangent lines of the same surface. The tangent lines are proportional to the lines of a surface in Euclidean geometry. Another model is the Poincaré disk model. The Poincaré disk is a model for hyperbolic geometry in which points are the points of the disk's surface and lines are represented as arcs of circles who are perpendicular to the disk's circle or as diameters of the disk's circle. Two arcs which do not meet correspond to parallel rays, while arcs which meet orthogonally correspond to perpendicular lines. We can find many applications for the results of Hyperbolic geometry, which range from the study of the shape of the universe as far as drawing.

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STUDENT PRESENTATIONS IN SCIENCE

SP4. NEUROLOGICAL DISEASES OF MANKIND AND ANIMALS: AN INSIGHT TO THE SOLUTION OF THE UNSOLVED

Seonhu Jeon, Tahee Strein
International School of Moscow, Russia

ABSTRACT

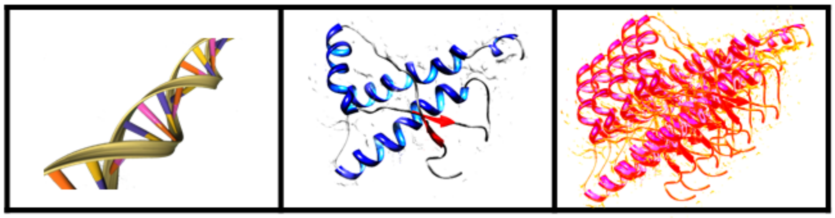
The advances in medicine have consistently allowed humanity to find creative solutions to previously unprecedented illnesses. The earliest recordings of its origins being traced back as early as before any constructed civilisations. Since the rapid modernisation of medicine from the 19th century onward, scientific developments have become increasingly frequent, and are still making rapid improvements today to the quality of healthcare and its broadened accessibility.
However, despite such efforts, the risks in the exploration of new concepts unavoidably receive criticisms and consternation as well as providing solutions to newly discovered diseases. There is still a vast majority of chronic illnesses that apply to both humans and animals that medical practitioners cannot "cure", simply because of their complexities that cause complications in different individuals. This is the limitation of the current course of known medicine, and it provides further enquiry and research to be done in the future, as it is a continual process in efforts to surpass such impediments.
Perhaps the progressive nature in the pursuit of improved and qualified medicine to cure the sick, whether it be humans or animals, is where the heart of this presentation lies. In the exploration of common, and other rather uncommon diseases, we will investigate possible solutions for a prion disease, Bovine Spongiform Encephalopathy (BSE), with its unidentified causes, like most prion diseases. The research will be conducted based on current solutions to similar neurological diseases, on both humans and animals, to broaden the research spectrum that will lead to our conclusion.

PRION PROTEIN

A prion is an abbreviated term referring to a 'proteinaceous infectious particle' that consists of proteins, without any genetic material such as nucleic acid genomes. Normal prion proteins (PrPC) are usually found on the surface of the cell membrane with a high affinity for copper ions. However, the entirety of the protein's functions is rather complex and needs to be further investigated.

Recently, Scientists have established the structure of a prion protein in both its normal and deviant forms, which are manufactured by E. coli bacteria that were altered through recombinant DNA techniques. Using magnetic resonance imaging and x-ray crystallography helped the researchers to understand key structural elements that allow the prion to co-opt the normal cellular form into the disease-producing variant.

Prions that are misfolded cause even more nearby prions to be formed abnormally, causing a chain reaction. Therefore, the stack of abnormal proteins propagates the disease and generates new infectious material. However, there is an uncertainty behind the reason for what is a cause of the prion's abnormality. These sets of proteins lead to various prion diseases like Bovine Spongiform Encephalopathy.



PRNP is a gene in your DNA which encodes for prion protein

Prion protein or PrP is a protein on the surface of your cells

A prion is an infectious particle made up of misfolded prion proteins

Artist's rendering; protein images adapted from Ilc et al 2010 http://goo.gl/WWlBR7

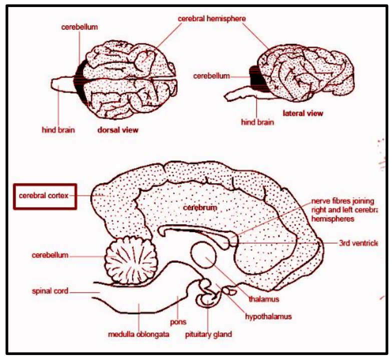
Prions can be transmitted, possibly by consuming abnormal prions and certainly by inoculation, either directly through the brain or through the muscle tissue. On the other hand, sporadic cases of prion diseases are rarely found in older age groups such as the middle ages or in the elderly due to the spontaneous chance of conversion from normal prions to deteriorated prions. Moreover, inherited cases may result from mutations in the PrP gene, leading to an alteration in the amino acid sequence of the protein. All prion diseases involve deterioration of the brain and other neural tissue, and as the treatment is still under development, infection from these diseases is fatal.

ANIMAL PRION DISEASES

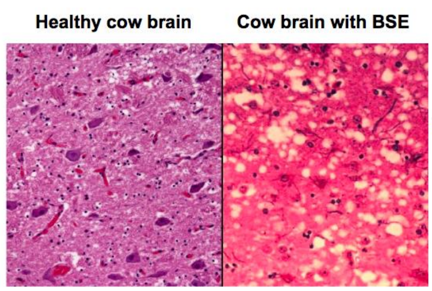
Prion diseases affect both humans and animals. Notable forms of animal prion diseases occur as Bovine Spongiform Encephalopathy (BSE) in cows, Chronic Wasting Disease (CWD) in deer, as well as Scrapie found in sheep. The diseases could be either sporadic, occuring at irregular intervals, or they could be due to genetic factors such as infected animals carrying mutant prion alleles.

BOVINE SPONGIFORM ENCEPHALOPATHY (BSE)

BSE is a progressive neurological disorder found in cattle due to an unusual transmissible agent called prion. The exact origins of the disease cannot be found, but the cause of BSE is due to feeding cattle meat-and-bone meal that contains tainted tissue from BSE or scrapie infected animals. As it is a very slow developing disease, the cows may not show any typical signs from 3 to 6 years. Fortunately, it is not a contagious disease and therefore the abnormal prion protein is not found in any dairy products.

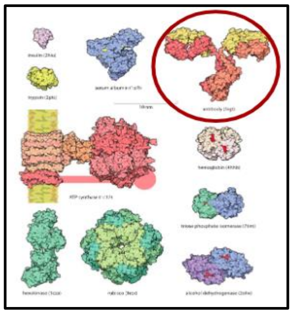


Most normal prions are found in the cerebral cortex, which is the outer layer of the cerebrum, composed of folded grey matter and this plays an important role in consciousness. According to the most widespread hypothesis, many tiny holes due to prions appear in the cortex, causing the brain to appear like a sponge.

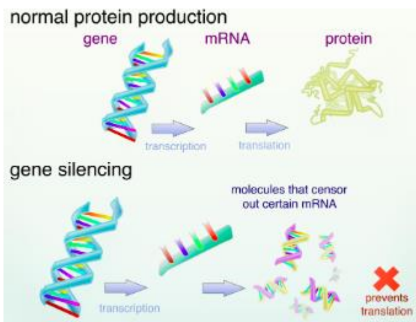


The abnormal prions are introduced by consuming tainted tissue and thereafter, this tissue enters the cortex. Gradually, the abnormal prions accumulate in the brain and cause even more nearby original prions to change to the abnormal form. At one point, the proteins begin to solidify, this is the stage where the cows are infected by BSE. Unlike other proteins, prions are very tightly bound so it is difficult to eliminate them through surgery. Once the cows are infected by BSE, as the name the 'mad cow disease' suggests, they may show: aggressive behaviour, difficulty in standing, weight loss and finally a decrease in milk production.

POSSIBLE TREATMENTS



One of the few possible treatments is the use of antibodies, which are proteins produced by lymphocytes in the presence of a specific antigen. As the size of a protein is very small, antibodies are larger in size. Due to their size, the antibodies cannot pass through the blood-brain barrier, which separates the blood from the brain and other extracellular fluids. To support this statement, a test on mice was conducted in 2003, where the antibodies were injected prior to the infection and, as a result, the mice did not show any severe symptoms of the disease. Conversely, when the antibodies were injected after the infection, the disease could not be cured as the antibodies could not reach the brain.

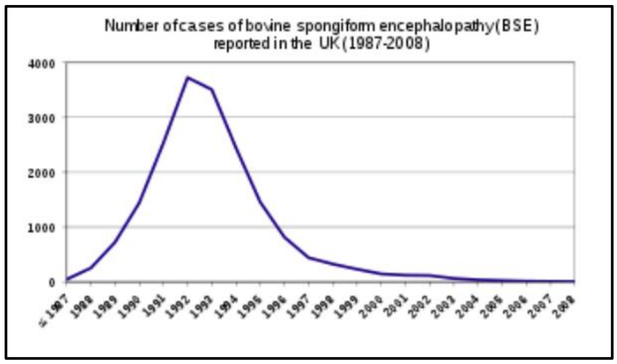


An alternative method would be gene splicing. This involves making tiny pieces of DNA or RNA which could stick to the prion producing RNA. In general, RNA is a nucleic acid that acts as a messenger, carrying instructions from DNA for controlling the synthesis of proteins. siRNA, which stands for small interfering ribonucleic acid, interferes with the translation of proteins by binding and promoting the degradation of mRNA at specific sequences. Thus, siRNA prevents the production of specific proteins based on the nucleotide sequences of this corresponding mRNA - the process is known as siRNA silencing. Compared to other treatments, gene splicing is one of the more recently developed treatments and has several limitations that need to be investigated further. Gene splicing is rarely used for medical purposes, as the treatment is a genetic manipulation that cannot be conducted on humans currently. Moreover, RNA and DNA are large in size, so they cannot readily pass through the blood-brain barrier. To support this hypothesis, an experiment was done regarding prion disease infected mice, where the RNA was injected to examine the changes in the damaged brain. Unfortunately, only a small area was impacted. This did however increase the survival rate by 18%.

In 2006, a research paper was published from the World Intellectual Property Organization (WIPO), suggesting the possibility of using the 'HIF 1-alpha protein' as an active ingredient in composing pharmaceutical and veterinary medication for preventing and treating prion diseases (PrP). The essential functions of the protein are listed as such: alleviating cytotoxicity of abnormal prion proteins against the nerve cells, hence inhibiting nerve cell death, and increasing the production of mRNA that largely composes the formation of new normal PrP. The compositions of the HIF 1-alpha protein would be preferably formulated in oral, inhalant or in dosage forms. However, issues regarding safety may be contemplated when achieving hypoxic conditions to activate the HIF 1-alpha proteins, since the side effects remain unknown.

Subsequently, an alternative solution may potentially arise from the research of Stem Cell Therapy. All somatic cells are specialised in their functions, and some somatic cells, such as neurons, are highly specific, non-mitotic or have long cell-cycles. The issue arises when these cells are severely damaged, the regeneration of these cells is nearly impossible. Stem cell therapy research provides a solution by substituting non-specific cells to perform other cellular functions either by replication or alteration to other cell types. The potential in this area of research is relatively high, providing multiple ways to compensate for the loss of neuronal cells caused by the abnormal PrP, as well as allowing replication of normal PrP.

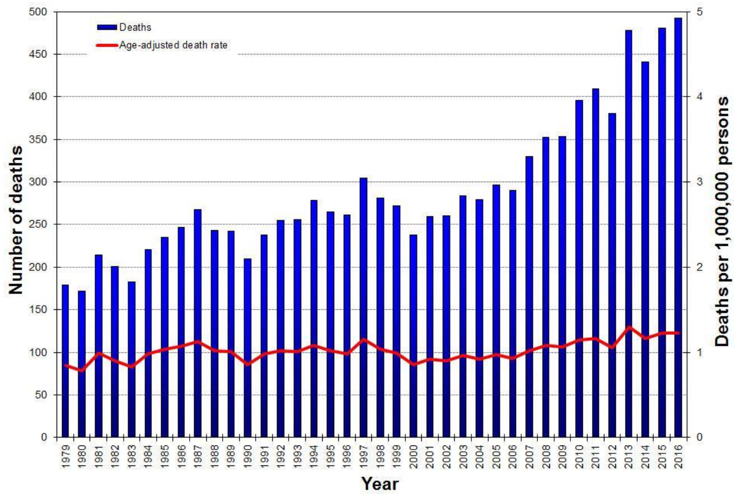
PREVENTION OF BSE



The vast majority of BSE cases, almost 99% in 1999, were reported in the UK. Subsequently, many people died from a similar neurological disease called the Creutzfeldt-Jakob disease. In order to prevent the spread of BSE, the American and Canadian governments began to restrict imports of live ruminants and certain dairy products. In the UK, ruminant feed was a crucial factor in the BSE outbreak and because of the current evidence for possible transmission to humans, in 1997 the US Food and Drug Administration (FDA) officially banned the productisation of bone meal and using cattle aged over 30 months as food. In 2001, the Department of Health and Human Services (HHS) issued an action plan, outlining steps to improve the scientific understanding of BSE.

CREUTZFELDT-JAKOB DISEASE (CJD)

A prion disease that is commonly known to occur in humans, may take the form of Creutzfeldt-Jakob Disease (CJD). The symptoms are very similar to those present in the elderly, who had formerly been diagnosed with other neurodegenerative diseases such as the Alzheimer's Disease and Huntington’s Disease. The symptoms include: dementia, hallucinations and confusion, as well as unsteadiness in physical activities caused by the reduction of neuronal cells. The causes of CJD remain sporadic, however the consumption of contaminated beef with diseases such as BSE is believed to increase the likelihood of contracting the disease. There has been a former instance during the 1950’s, where extractions of pituitary hormones from cadavers were injected as growth hormones in children. Unfortunately, the children were diagnosed with CJD shortly after their deaths, due to the transmission of abnormal prion proteins during the process. Additionally, any people with a familial history of CJD diagnosis, caused by genetic mutations in normal PrP making them act abnormally, are at a higher risk in developing the disease.

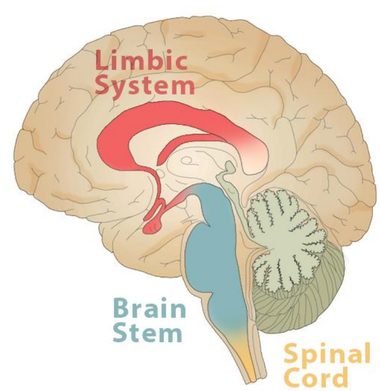


The first ever reported case of CJD breakout was in 1920, and from thence forth there has been a gradual annual growth in the number of reported deaths caused by the CJD. Economical and social factors may perhaps provide an explanation for such a trend. From the late 20th century, globalisation of the economy has lead to an expansion of trade, extending the accessibility of many products for consumers. One of the products that is still in high demand is meat, including beef, leading to an increased risk in exportation of contaminated beef with BSE.

However, despite the correlation between the economical factors and the number of cases reported of CJD, perhaps it would be more reasonable considering these factors as contributing to the number of causes, rather than presuming the causes are only derived from financial causes.

MEDICATION

Adequate medication for CJD is difficult to find, due to the causes being sporadic, and the medication currently available is mainly for pains and symptom alleviation. Specific drugs such as *Clonazepam* and *Sodium Valproate* reduce muscle spasms by moderating nerve cell movements. However, multiple side effects, especially on the level of mental health, may be of concern leading to the development of anxiety, depression, etc. Opiate drugs are also commercially prescribed as painkillers, as they alleviate pain and promote relaxation. The areas of our brain where opioid receptors are most present are in the limbic system, brainstem and the spinal cord, allowing the opiods to function as an effective anesthetic in reducing neuropathic pain.



As effective as the drugs may be, measures of precaution must be drawn due to the highly addictive nature of the opiate drugs. Chronic use of the opiate drugs, especially for long term illnesses such as CJD, may cause the patient to exhibit withdrawal symptoms of mild to extreme physical and psychological dependency. It therefore requires an expert or a medical practitioner to provide an appropriate prescription to patients in order to minimise these side effects.

PREVENTION OF CJD

Primarily, the most ideal and effective method in the prevention of the CJD would be not to receive organ transplantations or blood transfusions from people positively diagnosed with CJD or vCJD. Thus, a personal clearance beforehand for these procedures would be obligatory. Additionally, imported beef from countries where BSE is known to occur most, such as the UK, would be subject to quarantine. In hospital environments, decontamination of surgical instruments would be vital through various chemicals and autoclaving methods before subjecting the aforementioned instruments to a washer cycle, to proceed with routine sterilisation. Such efforts would significantly reduce the chances of CJD epidemics, as well as other diseases, occuring in hospital environments.

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SP5. THE EFFECT OF RADON AND RADIOACTIVITY

Lambros Lamprianou
The Junior & Senior School, Cyprus

ABSTRACT

This paper is about the effects of radon on human health. It has been argued that radon can cause serious health damage and could be associated to the disease that has been unfortunately running rampant in recent years; lung cancer. Radon is mostly associated with lung cancer. It is found in soil; even rocks in our houses give out radon, which escapes through cracks in walls and floors. Radon concentrations are usually highest in the basement or small spaces and corners. It is extremely important not to be exposed to high levels of radon for long periods of time. In this paper I discuss ways to avoid too much Radon concentrations. I also suggest that schools should be screened for the presences of Radon.

THE EFFECT OF RADON AND RADIOACTIVITY

This paper is about the effects of radon on human health. It has been argued that radon can cause serious health damage and could be associated to the disease that has been unfortunately running rampant in recent years; lung cancer (cancer.org, 2018).

Radon is associated with mostly lung cancer (Torres-Durán et al., 2014). Scientists estimate that about 20,000 lung cancer deaths per year in USA and 16% of all lung cancer deaths in Canada are related to radon (Evans, 2015; Chen et al, 2012). Radon gas is found in soil; even rocks in our houses give out radon, which escapes through cracks in walls and floors. Radon concentrations are usually highest in the basement or small spaces and corners. So, many people are exposed to radon just by being in basements or their own homes.

Radon gas itself is not lethal, but since it is a gas, we breathe it in. Once it reaches in and is exposed to the conditions of our lungs, it breaks down into other deadly elements that damage the lungs and lead to lung cancer. There is almost no way to save the person affected which is why it is extremely important not to be exposed to high levels of radon for long periods of time.

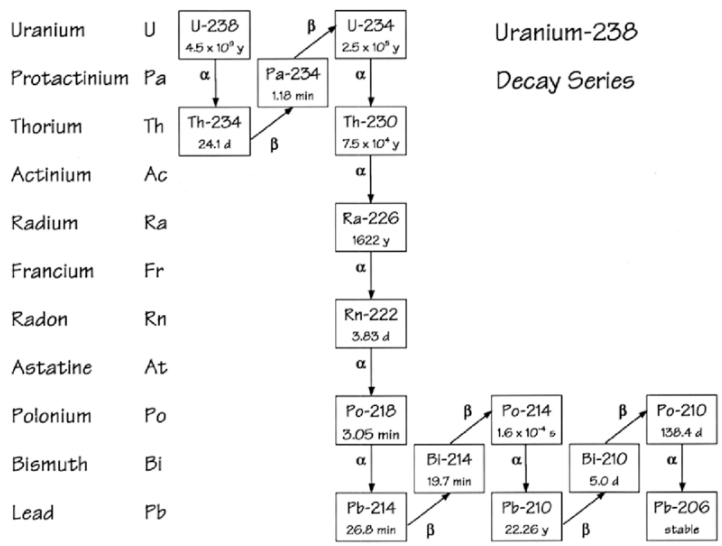
Radon and its properties (Introduction to Radon)

Radon is found in group 8 periods 6 of the periodic table making it quite unreactive. Radon is a gas at room temperature, becoming a liquid at -61.7oC and a solid at 71.15oC). Additionally, the gas is colorless and odorless.

Its massive relative mass number (222) compared to its smaller atomic number (86), suggests that Radon contains way more neutrons than protons in its “average nucleus”. This justifies Radon being a radioactive element.

Formation of Radon

Radon is part of the natural decay chain of Uranium-238 decay chain. Uranium-238 is found in rocks deep underground (most commonly found in Uranium ore). Uranium-238 has a half-life of 4.5 billion years. It decays in throrium-234 via alpha decay (look on the diagram, to the left) which then decays to protactinium-234 via beta decay then to Uranium-234 via beta decay, then to Thorium-230 via alpha decay.



Thorium-230 will eventually decay to Radon-226 then to radon-222 (both via alpha decay).

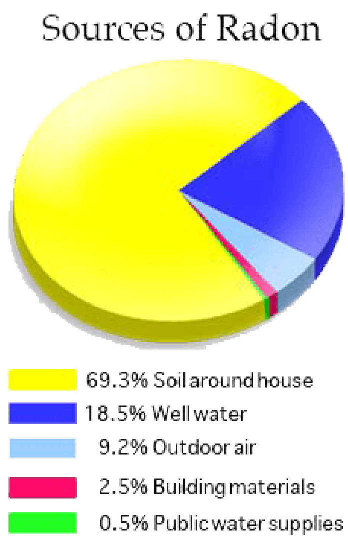
Alpha decay is the process in which an element irradiates a Helium nucleus or alpha particle (2 protons and 2 neutrons) from its own nucleus, reducing its mass number by four and its atomic number by 2. Alpha particles are very densely charged (with a charge of overall +2 due to of the 2 protons), hence they react quickly with particles around them, which, most of the time does not allow them to penetrate through paper or our skin. Also, they react with particles in the air as well, making, most alpha particles have a low range before they eventually give out all their energy and "die".

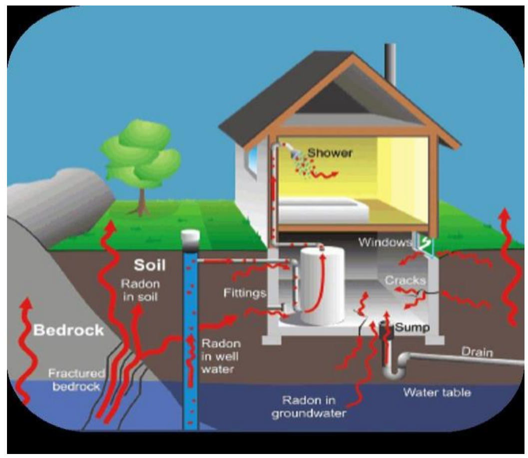
Why Radon is so dangerous

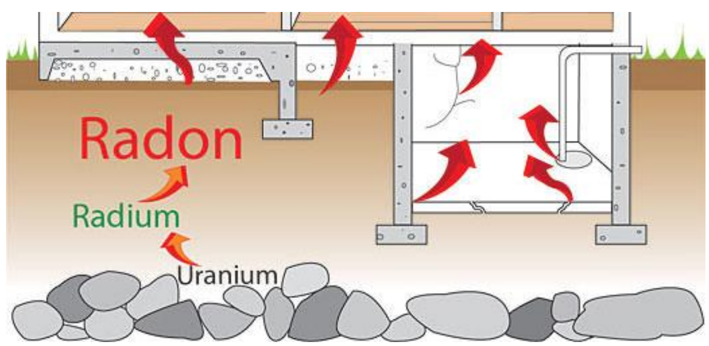
As mentioned previously, Radon is part of the natural decay chain of Uranium-238. As you can see from the diagram above, Radon later on decays using alpha decay into other radioactive elements, for example polonium. Radon itself is not dangerous to us, since alpha particles cannot penetrate through our skin, except in very rare cases. However if we ingest Radon, then the alpha particle can come in contact with our internal cells. The alpha particle, due to its high charge, will cause the DNA in our cells to change since it will react with the chemicals, such as the codons, which in turn causes the cell to mutate. This is later heightened as Radon decays into even worse radioactive products such as polonium-218, which can cause even more damage than Radon's alpha decay.

How we ingest radon

Radon comes from the decay of Radon, so it makes sense that Radon will be commonly found in underground rocks with high concentrations of Uranium. Radon being a gas means that it will make its way over the rocks and into our soil. Sometimes, it can even make its way above ground and enter our houses, where high concentrations of Radon will be built in small spaces such as small corners or small air spaces under doors.







The effects of Radon on our health

Once we ingest Radon, since it is a gas, it enters our lungs. There, it will start decaying and irradiating alpha particles which will damage the walls of our lungs. Then it will eventually break down in its other daughter nuclei (the elements it decays into) and cause even more damage and mutations to the cells in our lungs. If high concentrations of Radon are ingested, it is very likely that the person breathing in the Radon will suffer from lung cancer. Unfortunately, by then it is too late and there are almost no treatments that can be made, since the Radon and its daughter nuclei remain in our lungs. Continuously killing and mutating our cells. Scientists and studies from the World Health Organization estimate that about 20,000 lung cancer deaths per year in USA and 16% of all lung cancer deaths in Canada are related to radon. Lung cancer risk rises 16% per 2.7 pCi/L (100 Becquerel's per cubic meter) increase in radon exposure. In fact World Health Organization, 2009 studies show that radon is the primary cause of lung cancer among people who have never smoked.

Protective measures

Since there is nothing we can do after the damage is done, it is crucial we avoid inhaling high concentrations of Radon at all times. In many countries, people dig below the foundations of the house and install what they call a "radon sump". Radon sumps are pipes connecting the area below the foundations and the area outside the house. Radon in the ground underneath the house is forced into a fan which will blow air into the pipe and outside preventing it from entering the house.

What Could the government do

In some countries, governments have ordered for measurements for Radon levels to be taken in highly populated areas, or areas where schools or other buildings are to be built. If there is too much Radon, the construction will be cancelled and the whole area "quarantined" (no buildings can be built there in the future).

As a general rule of thumb, 4 pCi/L (148 Bq/m3), is where action has to definitely be taken either to reduce Radon levels, or avoid construction in general.

Although radon levels in Cyprus are generally low (Soteriadis et al., 2016), the Ministry of Education and Culture in Cyprus should follow the good practice of other countries and investigate schools for high levels of radon.

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SP7. DRIVING ON A RAINY DAY

Datis Kalali
Forum Private Greek School, Nicosia

ABSTRACT

Highway and road accidents are frequent during rainy days. In fact, in 2018 there were three consecutive accidents on the Nicosia-Limassol highway in one day, due to heavy rainfall. This project aims to calculate the maximum speed at which drivers can drive through highway curves on days with heavy rainfall and explains why a car's speed should be lower than the average speed limit on such days. Using basic laws of mechanics (circular motion), we can derive a formula for the maximum speed in terms of the gravitational constant g=9,81 ms^{-2}, the radius of the curve and the static friction coefficient between a car's tyres and wet asphalt. During the research, the static friction coefficient was calculated through an experiment. Moreover, using basic mathematics and an image of the road from the Global Positioning System (GPS), we calculated the radius of a curve in the Nicosia-Limassol highway. Hence, using the derived formula, the safety speed limit on a rainy day is 90 km/h, which is indeed less than the speed limit of 100 km/h on the highway.

INSPIRATION

In November 2018, three distinct traffic accidents were reported on the Nicosia-Limassol, due to heavy rainfall and two of the accidents occurred in the exact same area. This area (near Kofinou, Larnaca district) is curved and thereby, a possible reason could be that the standard speed limits of 100 km/h are high for driving on a rainy day and thereby drivers lose control of their automobiles. The aim of the research was to test this assumption and consequently the speed limit was calculated through a Physics experiment.

THEORETICAL BACKGROUND

The motion of a car on a road curve is an example of circular motion (i.e. motion of a body on a circular path), since the curve can be viewed as an arc of a circle. We assume that the car is driving at constant speed. In other words the magnitude of the car's velocity is constant.
Referring to diagram 1.1, it is evident that the direction of velocity in circular motion constantly changes, as the velocity vector is always tangent to the circular path. Therefore, in spite of the fact that the velocity's magnitude is constant the velocity vector does not remain constant. In other words, the motion is accelerated. This phenomenon can be explained using the concept of the centripetal acceleration $\overrightarrow{a_c}$, which is a vector with direction towards the centre of the circle and magnitude $\frac{v^2}{r}$ where r is the radius of the circular path and v is the velocity of the body. The

centripetal acceleration only changes the direction of the velocity and does not change the magnitude of the velocity.
When a motion is accelerated, it means that a force is acting on the body that accelerates or decelerates it. In uniform circular motion, this force is called the centripetal force (F_c) and is equal to the resultant force with direction towards the centre of the circle. Since this force corresponds to the centripetal acceleration, by Newton's second law of motion its magnitude equals to: $F_c = ma_c = \frac{mv^2}{r}$
Now consider a car moving on a curved path. Note that the car does not slide on the road so the friction between the tyres and the road is static. Referring to diagram 1.2, it is clear that this static friction acts as the centripetal force. So:
$|f_s| = |F_c|$
$\Rightarrow |f_s| = \frac{mv^2}{r}$
It is known that the static friction is always less than (or equal) the maximum static friction, and when the friction force acting on a body passes this maximum value, the friction becomes kinetic, meaning that the car will slide on the road and can possibly get out of lane and cause an accident.
So we must have:
$|f_s| \leq f_{smax} = \mu_s N,$
where μ_s is the static friction coefficient and N is the normal force applied on the car (which equals to the car's weight as the car is in equilibrium on the vertical direction)
Therefore
$|f_s| \leq \mu_s mg \Rightarrow \frac{mv^2}{r} \leq \mu_s mg \Rightarrow v^2 \leq \mu_s rg \Rightarrow \boxed{v \leq \sqrt{\mu_s rg}}$
The last inequality indicates that the maximum speed of the car must be $\sqrt{\mu_s rg}$ such that the car can be driven safely.

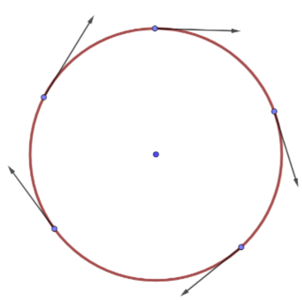

Diagram 1.1

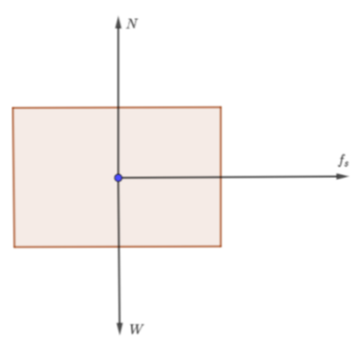

Diagram 1.2

CALCULATION OF THE FRICTION COEFFICIENT

In order to measure the friction coefficient between the tyres of a car and the ground the following apparatus was used:
A small piece of a car tyre, a piece of stone with similar textile to asphalt, a force metre, an electronic scale, a hook.
Initially, the mass of the tyre's piece was measured using the electronic scale. Then the piece of stone was placed stationary and a hook was connected to the tyre piece. The tyre piece was placed on the piece of stone and the force meter was connected to the hook. Subsequently, the force metre was drawn until the car's tyre started trembling and was about to move. The measurement on the force meter was recorded and the same procedure was repeated four times. Then again, the piece of stone was soaked in a sink of water for 1 minute and the removed without drying. The same procedure was repeated another four times.

Measurements:

Mass of tyre piece: 634g
Reading of the force meter (N) when stone was dry

4.31	*4.39*	*4.41*	*4.27*	***Average:*** *4.35*

Reading of the force meter (N) when stone was wet

1.35	*1.25*	*1.23*	*1.22*	***Average:*** *1.26*

Now theoretically at the moment when the reading of the force meter was taken, the friction between the two bodies was maximum static friction and the force applied on the body was equal to the friction since the body did not move.

Therefore $F = \mu_s N = \mu_s mg \Rightarrow \mu_s = \frac{F}{mg}$

Taking our average values for each case, we derive that when the stone was dry:

$$\mu_s = \frac{4.35N}{0.634kg \times 9.81\frac{N}{kg}} = 0.7$$

Moreover, when the stone was wet:

$$\mu_s = \frac{1.26N}{0.634kg \times 9.81\frac{N}{kg}} = 0.2$$

Other experiments have also proven that the static friction coefficient between a car's tyres and wet asphalt roads on rainy day is about 0.2 and can also get as low as 0.1. This supports the fact that the results of the above experiment do not contain high error.

CALCULATION OF A CURVE'S RADIUS

As for the experiment, a random curve was chosen in the Limassol Highway near the area of Moni.
A scaled picture of the curve was taken from the Global Positioning system (Google Maps), and using the computer the curve was estimated with a circle whose arc best represents the curve.

The radius of the circle in diagram 2.1, is 4.2 cm and the scale of the map is 2.5 cm for every 200 metres. Thereby it is evident that the radius of the curve is equal to

$$r = \frac{4.2cm \times 200m}{2.5cm} = 336\,m$$

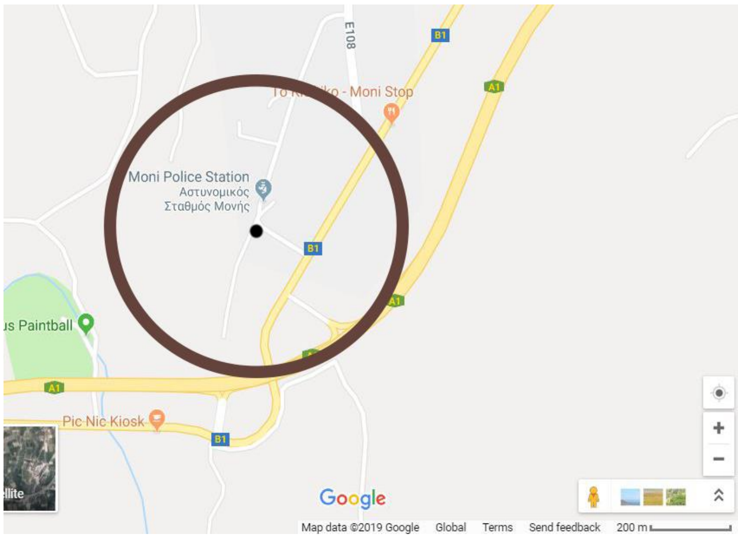

Diagram 2.1

CALCULATION OF THE MAXIMUM SPEED

Using the formula $v \leq \sqrt{\mu_s r g}$ we can say that $v_{max} = \sqrt{\mu_s r g}$
So using our measurements:

$$v_{max} = \sqrt{0{,}2 \times 336m \times 9.81\frac{m}{s^2}} = 25.7\frac{m}{s}$$

Therefore

$$v_{max} = 25.7\frac{m}{s} \times \frac{1km}{1000\,m} \times \frac{3600s}{1\,h} = 92.5\frac{km}{h}$$

Which means that $v_{max} \approx 90\frac{km}{h}$

RESULTS AND CONCLUSIONS

Now that the experiment was conducted, it is clear that a speed limit of 100 km/h is high for driving on a rainy day and will most probably cause automobiles to lose control and result in traffic accidents. In order to drive most safely on heavy rainy days, people should better drive at a maximum of 85 km/h to have full control over their vehicles.

ACKNOWLEDGEMENTS

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SP16. SPECTROPHOTOMETRY

Kakos Sotiris, Karali Alexandra, Kritharidis Konstantinos, Hadjicosta Marilena, Sfiris Haralambos Ioannis
The American College of Greece-Pierce, Greece

ABSTRACT

An important discovery concerning electromagnetism is that these waves consist of visible light, ultraviolet, X-rays, and gamma rays. Using this knowledge, after decades of extensive research and numerous experiments, scientists created a new scientific field related to electromagnetism, spectrophotometry. A basic aim is to determine what type of radiation absorbs a chemical substance. The basic principle is that each compound absorbs or transmits light over a certain range of wavelength. The spectrophotometer is the most common device used for that purpose. The first spectrophotometer, constructed by Arnold O. Beckman and his colleagues at National Technologies Laboratories in 1940, relied on using the amplifier from their pH meter, a glass prism, and a vacuum tube photocell. In this paper will be analyzed the principles of electromagnetic radiation, the history and the function of a spectrophotometer as well as the construction process of a simple version of it.

SPECTROPHOTOMETERS

Spectrophotometers today use light absorption to differentiate various colors and elements of substances at a highly accurate and precise level. Repeatability in color and composition can now be achieved on the processing line and from factory to factory around the world. Inter-instrument agreement of new technology ensures that no matter where a sample measurement is taken or who is operating the instrumentation, results will stay consistent and quality control standards will be met. The spectrophotometer is one of the most versatile tools used in today's modern science in light-based instrumentation and is used for various scientific and industrial applications. Spectrophotometers have infiltrated many various areas of science and manufacturing. They can be found anywhere from the field to warehouse to the production line and beyond. Spectrophotometric technology has found uses both inside and outside the laboratory. The spectrophotometer has multiple applications in the following fields and multiple historic examples exist: (a) Food Additives : During the World War II, scientists used spectrophotometers to identify vitamin A-rich foods to keep soldiers healthy as the role of vitamins was of significant concern, (b) Processed Foods: Identifying additives and chemical substances in general. They have various applications in water analysis as well, (c) Chemical Analysis: The DU spectrophotometer was used by scientists studying and producing the new wonder drug called penicillin, (d) The DU spectrophotometer was also used for chemical analysis of hydrocarbons and (e) Pharmaceuticals: wide applicability in molecular biology, studying photosynthesis and a wide variety of flowering plants and ferns.

These applications merely touch the surface of what light technology can achieve using spectrophotometric analysis. New research and discoveries are being made daily with spectrophotometric light technology.
We should mention that the spectrophotometer at its present form offers some quite important information about substances and elements in general. However, there might exist more things to discover and so improve the spectrophotometer based on those discoveries. Something worth mentioning is that probably any improvement to be made will be concerning mainly the use of the tool. Finding an easier and more efficient way to measure and find out about substances. Spectrophotometric technology advancements are constantly occurring throughout the industry. Even the instrumentation itself has changed immensely. Spectrophotometers are now smaller, more durable and lightweight, making them ideal for nearly any purpose or application.
As the list of applications grow, so do the needs for modifications in technology. Smaller and easier-to-use instrumentation is needed for portability and multiple user agreements. Ruggedness and durability are a must for in the field applications where instrumentation may be exposed to dirt, dust, and other elements of nature. As these modifications and various needs arise, light technology and spectrophotometric instrumentation must also adapt to these changes.

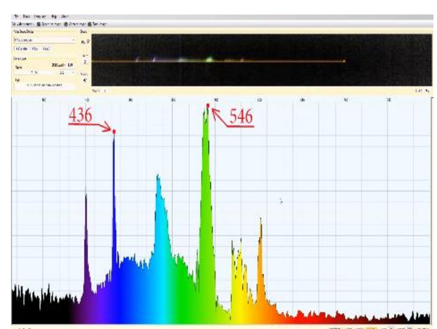
A BRIEF HISTORY OF SPECTROPHOTOMETRY

Beginning in 1940, the first in what became a series of Beckman spectrophotometers was developed through using the amplifier from a pH meter, a glass prism, and a vacuum tube photocell. Its creators were Arnold O. Beckman and his colleagues at the National Technologies Laboratories. This later became the Beckman Instrument Company. Commercial spectrophotometers had been developed some years before, but the two most popular instruments, the Cenco “Spectrophotometer” and the Coleman Model DM Spectrophotometer, did not have the capabilities to enable work at wavelengths in the ultraviolet.
The performance of the first spectrophotometer was not very satisfactory, but the design was quickly modified into the Model B version with the replacement of the glass prism with a quartz prism resulting in improved UV capabilities. It was followed with a Model C with improved wavelength resolution in the UV. No more than three Model C instruments were produced. The Model D instrument, also known as the Model DU, incorporated all of the electronics within the instrument case and featured a new hydrogen lamp with ultraviolet continuum and a better monochromator. This instrument retained the same design from 1941 until it was discontinued in 1976 after a commercial lifetime of 35 years. When initially produced, the Model DU was an immediate success because of its high resolution and limited stray light compared to other commercial spectrophotometers. By the end of 1941, 18 “Model DU” instruments had been sold for $723 and by the middle of 1942 another 54. When their production stopped in 1976, over 30,000 DU and DU-2 (a minor modification of the original DU) spectrophotometers had been sold. They were used in multiple scientific fields, including chemistry, biochemistry, and clinical and industrial laboratories. The Beckman Instrument Company grew to become one of the most important producers of a wide range of research and medical instruments. Nearly every biochemistry and clinical laboratory in the world owned, or had access to, a Beckman pH meter, a Beckman DU spectrophotometer, a Beckman analytical ultracentrifuge, and other Beckman instruments.

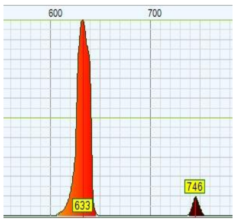
CONSTRUCTING A SIMPLE SPECTROPHOTOMETER

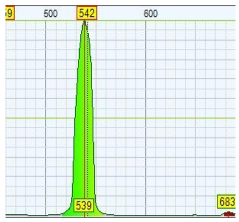
In this part of our project, we will see the construction of a simple version of a spectrophotometer step by step. It is a very simple procedure that requires a few materials and basic skills concerning mainly measuring and using cutting tools. Afterwards, we will calibrate and test our device. Thus, we will follow a procedure that both fits with our academic level and illustrates spectrophotometer's structure up to a significant point. Regarding the equipment needed, it is cheap and can be easily found inside a household. More specifically, the materials which will be used are a computer or laptop, a USB webcam, an empty DVD disk, a light source (i.e. a lamp, a torch etc.), a dark housing, a black cardboard, a ruler, a protractor, a pair of scissors, a cutter and both stick and hot glue. It is important to take care of the dark housing which is basically a midsize box whose interior needs to be black. To achieve better results it is preferable to make one by yourselves using either some cardboard or some wooden planks. In addition, it is, also, necessary to use specific software in order to analyze our spectra during the experimental procedure. Although there is a variety of such programs in the internet, our team suggests using one of the following. The first one is Spectral Workbench ("https://spectralworkbench.org/capture") which is online and the second Theremino ("https://www.theremino.com/en/downloads/automation#spectrometer") which requires downloading.
As far as the constructing process is concerned, it is neither tough nor time-consuming. First of all, we cut a small slit in the side of our box which does not need to be precise. Then, we take the cardboard, that is smaller than our housing mechanism and fits in it, and we fold its edge up about to 3 cm along the side of our box. Now, we need to take care of the following cut. We take the edge of our cardboard and we make a slit which is 1mm in width. As we affix it in the box, it is important to match up with our previous bigger slit. Afterwards, we cut a little piece of the DVD and try to split it in half carefully. We keep the non-coloured part that is going to separate the lens from our webcam. Next, we cut it in order to form a little rectangle which is going to fit right over the webcam. Finally, we connect the webcam with our computer, we put it inside the box across the split in a 45 degree angle and our device is ready. In this stage, it may be necessary to manipulate specific parameters meaning to change a bit the place of the DVD disk or the camera in order to achieve the best possible depiction of our light source's spectrum.

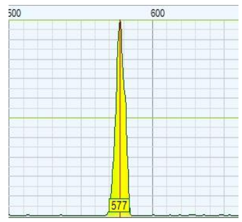
Last but not least, we need to calibrate our device. This is done by following some simple steps. First of all, we put across the slit a regular fluorescent lamp and try to find its 2 stable peaks. Then, we click tools, trim points and we choose "fluorescent 436 and 546". After that, on the top of the diagram a scale is appeared according which we try to align the peaks respectively. Finally, we click trim scale at the bottom of our page and the calibration is done. However, since we did not find that lamp, we calibrated our device with a white night lamp following the same steps.



Therefore, we are about to test our spectrophotometer. There are multiple experimental procedures that can be followed. Our team chose to tested using red and green laser pointer and the light emitted by Sodium chloride (solution)-NaCl(aq) when it is burnt.





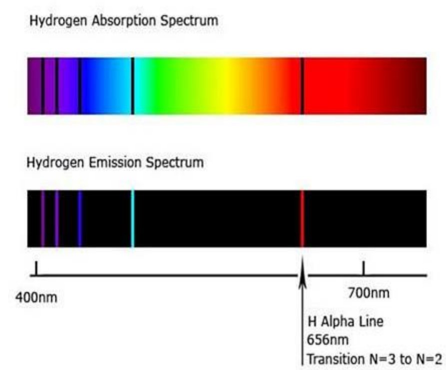


Firstly, the expected measurement of red laser pointer is at approximately 635 nm when ours is at 633 nm. Moreover, regarding the green laser pointer the expected is at 532 nm which is close to ours at 539 nm. Lastly, the expected measurement of Sodium Chloride is at 590 nm - 13 nm away from ours at 577 nm. Therefore, as it is evident our measurements are pretty close to the expected ones meaning that our device is ready to be further used. However, since the measurements we got were not identical to the ones expected according to our bibliography, we realise that limitations existed and ways to improve our spectrophotometer must be found. Firsty a major limitation is the fact that we did not calibrate the instrument using a regular fluorescent lamp as the Theremino developers suggest. Now, as far as the improvements are concerned, we think that it would be better to enable our camera to detect infrared range. Furthermore, we could manipulate specific parameters such as the box shape and size, the diffuser's material or test different USB cameras and programs. Finally, we think it is important to elaborate techniques which will make possible the use of such devices in school chemistry labs.

ELECTROMAGNETIC RADIATION

Electromagnetic wave is the simultaneous propagation of an electric and a magnetic field. Electromagnetic waves propagate in the vacuum at the speed of light. In all materials they propagate at a slower speed. The electromagnetic wave is transverse. The vectors of the electric and magnetic fields are perpendicular to one another and perpendicular to the propagation direction of the wave. The cause of electromagnetic wave generation is the accelerated movement of electrical charges. The light emitted by the stars is part of the overall spectrum of electromagnetic radiation that is found in the universe. The electromagnetic radiation, according to the frequency of its waves and respectively the energy it transports, is divided into regions. These are radio waves, microwaves, infrared, visible light, ultraviolet rays, X-rays, and gamma rays. All these forms of electromagnetic radiation move at the speed of light and can even penetrate certain materials. The forms of radiation that don't penetrate the material, ie. get absorbed by it, vary depending on the composition of the material. The absorbance of an object quantifies how much of the incident light is absorbed by it (instead of being reflected or refracted). This may be related to other properties of the object through the Beer–Lambert law. Precise measurements of the absorbance at many wavelengths allow the identification of a substance via absorption spectroscopy, a technique, according to which, a sample is illuminated from one side, and the intensity of the light that exits from the sample in every direction is measured. A few examples of absorption spectroscopy are ultraviolet–visible spectroscopy, infrared spectroscopy, and X-ray absorption spectroscopy. Absorption spectroscopy refers to spectroscopic techniques that measure the absorption of radiation, as a function of frequency or wavelength, due to its interaction with a sample. The sample absorbs energy, i.e., photons, from the radiating source. The intensity of the absorption varies as a function of frequency, and this variation is the absorption spectrum.

A material's absorption spectrum is the fraction of incident radiation absorbed by the material over a range of frequencies. Radiation is more likely to be absorbed at frequencies that match the energy difference between two quantum mechanical states of the molecules. The absorption that occurs due to a transition between two states is referred to as an absorption line and a spectrum is typically composed of many lines.
The frequencies where absorption lines occur, as well as their relative intensities, primarily depend on the electronic and molecular structure of the sample. The frequencies will also depend on the interactions between molecules in the sample, the crystal structure in solids, and on several environmental factors (e.g., temperature, pressure, electromagnetic field). The lines will also have a width and shape that are primarily determined by the spectral density or the density of states of the system.

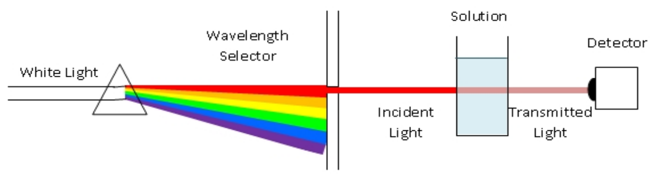


A FUNDAMENTAL USE OF A SPECTROPHOTOMETER

Spectrophotometry is a method to correlate a chemical substance's absorbance or transmission of light with its properties. The basic principle is that each compound absorbs or transmits light over a certain range of wavelengths. It is a field of science with two principal functions, depending on the type of spectrum that is examined. It is divided into Spectrophotometry of Emission and Absorbance.

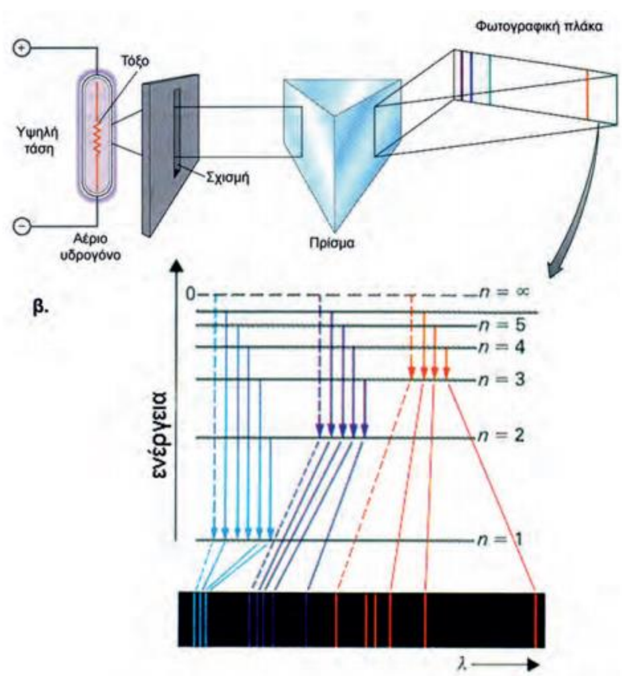
First of all, to understand the procedure of the Spectrophotometer it is important to analyse the structure of the atom. The atom, as scientist Niels Bohr described it, consists of the nucleus and the electrons spinning around it. The electrons spin in certain orbits called electron shells. Each shell corresponds to a certain amount of energy. It is also important to mention that each element has a different number of shells and different energy levels. The electron has the ability to "jump" to a next shell by increasing its energy. That happens when the atom gets in a state of excitation, by getting heat energy, electric energy, or by absorbing radiation. Radiation is, according to Max Planck, quantized, which means that it is emitted into small "packets" of energy called photons. To get to the next energy level the electron must receive the exact amount of energy that is equal to the difference of the energy of the next shell minus its current energy. The state of excitation lasts for fractions of seconds and then the electron returns to its initial position while either emitting the energy it got, or absorbing it. That is the basic difference between spectrophotometry of emittance and absorbance.

The structure of the Spectrophotometry of Absorbance is simple. A light source emits light that is later split by a slit, in order for only one spectrum of light to reach the solution. The incident radiation travels through the sample, a part of it is absorbed by the substance and the transmitted light then reaches the detector. The result is the absorption spectrum, the fraction of the incident radiation that was absorbed by the material. The absorption spectrum is unique for each element, because of the different frequencies, electronic and molecular structures and environmental factors. That is why the absorption spectrum of an element can be characterized as its fingerprint. One of its functions then, is to recognize the element used. The structure of the Spectrophotometer of Emittance is different. The substance under study gets energy and thus its atoms get in a state of excitation. It then emits light, the radiation that corresponds to the energy it used to swift to the next energy level. That beam of light is separated through a prism and in the detector appears the emission spectrum. It is equally unique and is complementary to the one of the absorbance.



The difference between the incident and transmitted light indicates the absorbance

Another function of the Spectrophotometer is the application of Beer and Lambert's Law. It states that the absorbance is equal to the product of λ, a coefficient that is depended on the wavelength and thus the substance, d, the distance the light travels in the solution and c, the concentration, the quantity of the substance per 1 L of solution. A common application of Beer and Lambert's Law is the calculation of the concentration if the other factors are known.



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SP17. STEM CELLS

Arvanitaki Victoria, Gkerekou Georgia, Loukaki Vasiliki, Charalampos Papadopoulos-Siountris, Christina Servou, Antonia Stamatellou
The American College of Greece-Pierce, Greece

ABSTRACT

Stem cells are of utmost importance for all organisms to survive and most cells in our body come from them. They have the ability to reproduce and to differentiate to any kind of cells in the body thus replacing damaged or diseased cells. Stems cells can be found in any human body. More specifically, sources of stems cells are the placenta and the umbilical cord blood, the umbilical cord tissue, the primary dentition and the adipose tissue. Stem cells can be used for the cure of malignant diseases, especially in one's blood or for problems in the metabolism. Furthermore, it is possible to use them to restore the function of organs and bones that might not be working correctly either from accidents or aging. Also, many transplants in bone marrow have been done successfully. Other than the usual uses of stem cells they appear to be able to help in complex and various ways patients, research studies and even companies, corporations or governments. For example, advance treatments for cancer and genetic defects and even test new drugs for safety and effectiveness. It is vital for human stem cells to be preserved for future treatments and clinical trials. Today, this is achieved with two methods namely, cryopreservation and hematopoietic preservation. Using a specific web platform, we will study ways of differentiation in stem cells. Moreover, we will attempt to examine different parameters of transforming one type of stem cell to another. Ethical aspects of stem cells applications will be examined.

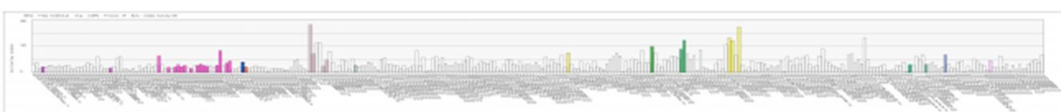
CURRENT USES OF STEM CELLS

Stem cells themselves do not serve any single purpose but are important for several reasons. They can take the role of another type of cell and are able, under the right conditions, to regenerate damaged tissue. Through this way many lives could be saved and wounds or tissue damage could be repaired. Many doctors have performed stem cell transplants that are also known as bone marrow transplants. Blood-forming stem cells in the bone marrow were the first stem cells to be identified and the first to be used in the clinic. Furthermore, in a stem cell transplant the first thing that needs to be done is the embryonic stem cells to specialize into the necessary adult cell type. Then, those mature cells replace the damaged tissue. Researchers are also testing adult stem cells to treat other conditions, including degenerative diseases such as heart failure. There's no limit to the types of diseases that could be treated with stem cell transplants as they have the potential to make breakthroughs in any of them. For instance, they are able to: (a) replace neurons damaged by spinal cord injury, Alzheimer's or Parkinson's disease or other neurological problems that a person might have, (b) produce insulin that could treat people with diabetes and (c) produce heart muscle cells that repair the damage after a heart attack. Nowadays, medicine goes forward, and the list of diseases, where the use of stem cells is a standard medical treatment is constantly growing.

Other than the usual uses of stem cells, like understanding how the development of a fertilized egg to a fully-grown organism works or treating diseases, they appear to be able to help in complex and various ways patients, research studies and even companies, corporations or governments. For example, they could create and replace damaged tissues and organs, help research the causes of cancers and genetic defects and even test new drugs for safety and effectiveness. Other than the usual uses of stem cells in science fields, there are also plenty of other ways these undifferentiated cells appear to be able to help in complex and various ways not only patients but research studies and even companies, corporations or governments.
Stem cell research is taking place at universities, research institutions, and hospitals all over the world. The focus for researchers at the moment is to find ways to manage how stem cells turn into other types of cells. While watching the stem cells mature into cells that can be found in many different places in human body, researchers and doctors may also better understand how diseases and conditions come to be in the first place. The safety and effectiveness of new medication can also be tested using differentiated stem cells. Testing drugs on human stem cells eliminates the need to test them on animals as well. For the testing of new drugs to be accurate, the cells must be programmed to acquire properties of the type of cells targeted by the drug. For instance, nerve cells could be generated to test a new drug for a nerve disease. Tests could show whether the new drug had any effect on the cells or whether the cells were harmed. Scientists from Wake Forest University of South Carolina (US) have managed to develop a tool that combines a basic stem cell use with one of the latest pieces of technology out in the market today. They are 3D-printing stem cells onto biodegradable scaffolds to make custom-made living body parts. The stem cells are printed in a hydrogel solution using a special 3D printer called "ITOP". This printer makes it possible for the printed stem cells to develop into life-sized tissues and organs that have built-in microchannels that allow blood, oxygen and other nutrients to flow through. Thus far the team was able to generate segments of jawbone, muscle tissue and even a whole ear. Stem cells are considered a promising therapeutic strategy for restoring sight in patients suffering from diseases of blindness. One study from UC San Diego Health (US) treated children suffering from cataracts. They removed the cataracts and stimulated the native stem cells in their eyes to produce new lens tissue that was able to improve their vision. Another study generated different eye parts in a dish using reprogrammed stem cells. After the procedure they proceeded into transplanting them into blind rabbits and were successful in restoring their vision. Hopefully soon stem cell technologies will advance through the clinic and provide new treatments to cure patients who've lost their sight.
Human stem cells have become one of the hottest areas in biotechnology as several companies have jumped in to try to exploit them commercially in medical therapies. Most of the companies are working with adult stem cells, which pose fewer ethical problems. Some of these companies like "Cellularity" or "Rubeus Therapeutics" are already moving their stem cell products into clinical trials, to use, for example, in restoring cancer patients' immune systems after intense radiation or chemotherapy or fighting against blood cancers. But no one yet knows which companies will actually manage to achieve their goals, bring something new to the table and help millions of lives across the globe.

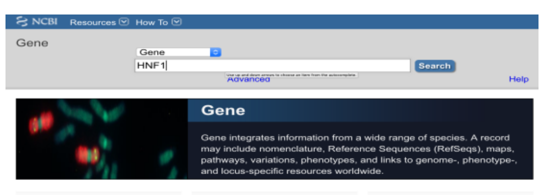
TRANSCRIPTION FACTORS

It is widely known that stem cells can be changed from their usual form to another stem cell by differentiating, but what is not known is that an important procedure before doing that is taking specific transcription factors. By doing that we are not only controlling the expression of specific cell types but also taking the correct measures for preventing a disaster. We will study how scientists provoke stem cell differentiation and how they can transform a stem cell into a specific type of cell. For example an ***x*** stem cell can become a neuron, a blastocyst, a red blood cell, cardiac cell etc. Now, in order to do that a scientist must follow specific steps, which will be analyzed later on. Continuing, in molecular biology, a transcription factor (T.F.) is a protein that controls the rate of transcription of genetic information from DNA to messenger RNA, by binding it to a specific DNA sequence. But, how do we select the most appropriate transcription factor? It is hard to find which transcription factor is the best for a cell. In order to find out the ones that fit, a bit of research is required. After finding information in different web sources, the following T.F. which were found are: HNF1, HNF4, CRX, GCNF, HNF3, PPARA, SP3, ER, TEL2 and finally SREBP. But a very important question that we need to answer is: Do these Transcription Factors control other tissue gene expression? To find out, we need to search some web-databases for that kind of information. After a long time, we concluded in the best website for this data, which is called '*Amazonia!*' and is a web-database for transcription factors (http://amazonia.transcriptome.eu/). Now, if the transcription factor we are looking for is the HNF1 a chart with it's gene expression regulation will come up in the middle of the screen (as shown) and will show every expression for genes that can 'work' with that Transcription Factor.



Chip : U133P2 - Protocol: SP - Norm.: Global Scaling 100

From the data that we have collected we found that HNF1 can control the expression of the following genes: *Oocytes, liver, small intestine and Peripheral Neuron System.* Moving on, it is crucial to know if this transcription factor relates to potential health issues?
To search these issues, we can search a database called: *The NCBI Gene Database.* In this database we will search any health issues that relates to e.g. HNF1. After pressing 'Search', many categories per species or per organisms will show up and the correct one will be *Homo Sapiens,* which is the human species. If you are doing this procedure it is important to be sure that you have found information for the species/organism that you are looking for.



After reading every information pay attention to the summary which talks about some potential health issues that relate to HNF1:

Summary The protein encoded by this gene is a transcription factor required for the expression of several liver-specific genes. The encoded protein functions as a homodimer and binds to the inverted palindrome 5'-GTTAATNATTAAC-3'. Defects in this gene are a cause of maturity onset diabetes of the young type 3 (MODY3) and also can result in the appearance of hepatic adenomas. Alternative splicing results in

This was one example. But there are many others if you make a more wide research.

Finally let's sum up how we choose the idealist Transcription Factors. Firstly, explore the demands for specific types of cells or other tissues. Then, seek for Transcription Factor which to control this tissue gene expression. Afterword's, seek for other tissue gene expression that is controlled by this Transcription Factors. Search, also, for health issues associated to this TF. And last but not least, assess all data.

STEM CELLS STORAGE AND PRESERVATION

Stem cells can be collected in three phases of one's life. Firstly, they can be isolated from an embryo at the second stage of prenatal development. These stem cells are called embryonic and they are pluripotent, which means that they are extremely powerful as they can become any cell type. Though with this source of stem cells there are some ethical issues. Two more sources of stem cells, the most common ones, are the cord blood and the cord tissue. Cord blood is the remaining blood of a baby's umbilical cord and placenta after it is born. Cord tissue is a part of the umbilical cord. The stem cells that are collected right after childbirth from the placenta and the umbilical cord are called perinatal and they are perfectly matched to the baby. They are, generally, more capable in differentiation that adult stem cells and they in comparison with the embryonic stem cells they are not related to any ethical concerns. Last but not least, stem cells can be found throughout an adult's body. They are called adult stem cells and they exist in the brain, skeletal muscles, blood vessels, blood, adipose tissue, dental pulp bone marrow, skin, and liver. Isolation of stem cells is a gentle and complicated process with little differences based on their source and kind. This procedure is taking place in specially designed sterile positive pressure labs (Cleanrooms). Those rooms follow 100% the very strict specifications which are also used in case of bone marrow transplantation. All samples are examined with automatic devices for microbial infections. The virological checkup is achieved with a method called PCR (Polymerase Chain Reaction). What this method does is that it makes a lot of copies of DNA fragments and in this instance, with stem cells, some genetic diseases or disorders can be traced by the genetic material of the virus even in small amounts of blood serum. In terms of examining the stem cells for bacterial or fungus infections, a special automatic blood culture system is used. The stem cells that will finally be selected are analyzed with an appropriate device in order to be counted the total number of cells with a nucleus. After that, with flow cytometry, a method used to spot and determine physical and chemical characteristics of a group of cells, is specified the population and the viability percentage of the cells. When this procedure is completed, the stem cells are ready for storage and preservation.

Stem cell storage is defined as the isolation and cryopreservation of those cells in order to be used in the future for treatment and clinical trials. The technology that is used to preserve and store stem cells can offer viable cells for approximately 25 years. A baby's stem cells can be useful for family members with a 1-in-4 chance of being a perfect match and for siblings with a 3-in-4 chance. For the owner, they are the perfect match and there is no possibility of rejection. The method that is used to preserve any type of stem cells is called cryopreservation. The purpose of this technique is to minimize the damage of the cells during freezing and storing at very low temperatures so that the cells would viable without lesions. For the cryopreservation to be successful the scientists use a process named “gradual freezing” which requires the involvement of special equipment, computers, and certain programs. It is important for stem cells to be gradually frozen because otherwise the water from which they are mostly made of would become ice quickly and they would die immediately. Though cryopreservation aims at repressing the function of the cells without destroying them. For that reason, the temperature must be dropping steadily but not too slow nor too fast with constant monitoring while a cryoprotectants substance has been applied. This whole process of gradual freezing guarantees the viability of the stem cells when the time comes to be defrosted and used. According to international directives stem cells are being stored in special bags designed for human cells protected in metallic cases. This bag has two or three smaller tubes which hold representative samples of the content of the bag. These samples are used for identification of the stem cells, and other tests. In addition, two vials are kept for testing, genetic analyses, and testing. Finally, all specimens are placed in individual containers which supply liquid nitrogen, at -150 Celsius. Temperature affects the viability of stem cells during storage and after the defrosting. As a result, the lower the temperature is the longer the profitable storing period lasts. As an extend, the liquid nitrogen riches from -150 degrees Celsius to -196 degrees Celsius. By maintaining these temperatures is certain that the majority of the cells are safely inactive.

STEM CELLS CHARACTERISTICS

As we know our body consists of many different cells which are responsible for specific functions. However, there is one category of cells that does not have a specific role and can become almost any cell type. These are called stem cells. They are divided into two main types: The embryonic stem cells (EScells) and the adult stem cells or tissue stem cells. The reason why stem cells are considered so important is because they have an essential role in therapies for many disorders and injuries. All stem cells have three unique characteristics: (a) they have no specialized functions, (b) they divide and renew themselves for long time and (c) they can differentiate and generate specialized cell types through the process of differentiation. According to their differentiation potential, stem cells are also divided into five categories: (a) Pluripotent: They are able to generate almost every type of cell. Embryonic stem cells are considered pluripotent, (b) Totipotent: These stem cells can form all different types of cells. The zygote is a totipotent stem cell, (c) Multipotent: They can turn into specific cell types which are closely related. Hematopoietic stem cells belong to this category since they produce only platelets and red and white blood cells, (d) Oligopotent: These stem cells can generate limited types of cells, like lymphoid and myeloid stem cells, (e) Unipotent: They have a limited ability to differentiate since they can only produce cells of their type. They can also self-renew.

There is also another very important category of stem cells which is called Induced Pluripotent Stem Cells (IPSCs). In 2007, two researchers, Shinya Yamanaka of Kyoto University and John Gurdon of Cambridge University managed to convert adult stem cells into cells that behave like embryonic stem cells. So, these cells can give rise to every other cell type in the body like embryonic stem cells do, but without having to face any moral issues. That discovery was so important that in 2012 they were awarded the Nobel Prize.

BIOETHICAL ASPECTS OF STEM CELLS USE

As discussed above, the stem cells taken from early-staged embryos can be very useful, as they can be subjected to a wider variety of differentiation and therefore be used for the cure of multiple diseases. Nevertheless, there are lots of ethical concerns being evoked by researches concerning those stem cells. That's because in order to receive the embryonic stem cells, scientists have to destroy the blastocyst and thus are killing the fertilized egg in the early days of its development. Therefore, the idea of killing a person in order to use his cells for the cure of someone else causes the justifiable aversion of some. However, others insist on the great results of the use of embryonic cells as lots of people's suffering, due to severe diseases, could end.

On the one hand, several views converge on the belief that the fetus in the early stages of growth is just a mass of cells and nothing more. On the grounds of that, it's believed that research should be allowed as the fetus could not survive outside of the uterus. Human cells used in laboratories are considered too rudimentary in development to have a moral status. Moreover, it has been proven that not all of the cells of the cell mass are exclusively destined to become part of the embryo itself (all of the cells contribute to the placenta too).

On the other hand, people who consider embryos created with preimplantation to be human beings generally believe that such work is morally wrong. According to this view, the fetus bears human status from the moment of its conception, and therefore no research action and use, that does not aim at its benefit, should be allowed. The divergence of views on this issue is illustrated by the fact that the use of human embryonic stem cells is allowed in some countries and prohibited in others. Due to that, the relevant legislation may vary across European countries. As a matter of fact, human embryonic stem cell research is allowed in England and Sweden, and is banned in Italy, Ireland, Germany. Other countries, including Greece, are allowed to obtain embryonic stem cells from very early embryos, under the condition that they are conceived through IVF (in vitro fertilization). Religions around the world also hold different positions regarding the matter. To begin with, both the orthodox and catholic churches believe that the body and the soul are formed at the same time (the moment of conception) and no physical milestones, such us the the development of the nervous system, are prerequisites for ensoulment. Consequently, they support that all fetuses should be respected and protected because of their potential to grow into perfect human beings. Under hinduist and buddhist teachings, an embryo acquires personhood after implantation in a mother's uterus so fetuses are considered to be human beings. Despite that, both religions are more flexible on that subject, if research is intended to help humankind. Lastly, Judaism values embryos' potential to become human beings, but does not accord them the same status as a person (the embryo is 'mere water' until the 40th day when the soul takes up residence in the body). Similarly, the Islam is also making the distinction between actual life and potential life and although the Quran doesn't specify things, a time point of 120 days after conception has been embraced. If a sufficing amount of pluripotent stem cells could be obtained from adult tissues, an important source of ethical conflict would vanish. However, when stem cells are obtained from human embryos, ethical objections arise from the need to destroy the fetus to obtain those cells. From a moral point of view, no one could give the impression to have the answers, nor the solutions as the questions raised are multiple and constantly being renewed. As a result, it is certain that in order to be able to form an opinion, we must consider all the different variables and concentrate on whether a use like this would have mostly positive or negative results on humanity.

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SP19. CHROMATOGRAPHY

Giagli Xenia, Papakosta-Sabatakaki Nelly, Papanikolaou Archono Rafaela, Papadakis Nikolaos
The American College of Greece-Pierce, Greece

ABSTRACT

The history of chromatography begins during the mid-19th century when a rudimentary version of the technique was used for the separation of plant pigments such as chlorophyll. Chromatography is a process by which substances can be separated from their mixtures. There are many types for each of which different materials are used but they work in the same way. The mixture to be analyzed is placed on one end of the material, called the "stationary phase" and is carried to the other end of the material by a fluid, called the "mobile phase". As some substances are more soluble in the fluid, they travel at faster speed, and thus their separation is achieved. Modern chromatography is applied in many fields of science, such as Biology, Medicine, Pharmacy, Chemistry, Agriculture, Toxicology, as well as in food science and industry. Chromatography, therefore, contributes to the development of the sciences all over the world and to the invention of many beneficial techniques. In this article, we will analyse the history, the basic principles, current applications and different categories of modern chromatography.

TYPES OF CHROMATOGRAPHY

Due to the variety of chromatographic techniques, their classification is impossible, using only one criterion. Thus, they can be assorted by the nature of the stationary and mobile phase, by the separation mechanism, by the form of the stationary phase and by the way of intake and movement of the substance.

The chromatographic techniques are classified as liquid Chromatography (LC) and gas chromatography (GC), depending on the nature of the mobile phase. These categories are further narrowed down, depending on the nature of the stationary phase, to liquid-solid chromatography (LSC), liquid-liquid chromatography (LLC), gas-solid chromatography (GSC), gas-liquid chromatography (GLC).

Based on the separation mechanism, the techniques are classified, based on the mechanism which retains some ingredients of the substance, to: ion-exchange chromatography, partition chromatography, molecular exclusion chromatography and affinity chromatography. In Ion-exchange chromatography, some ionic ingredients of the mix are retained from the opposite charged stationary phase, to a different degree. The balance between the retained and non-retained ingredients achieves the separation. In partition chromatography, the ingredients are allocated in a thin layer of the liquid stationary phase, which is formed on a solid inactive substrate. The liquid mobile phase will separate the ingredients based on solubility in the two liquids. In molecular exclusion chromatography, the stationary phase has porous, of specific size, which hold the molecules of that size. The molecules of larger or smaller size will pass through the stationary phase quicker, thus achieving the separation. Affinity chromatography, which is one of the newer techniques, is based on the specific interaction of a molecule of the substance with the stationary phase, resulting in the separation.

Based on the form of the stationary phase, the techniques are assorted in two categories. For the majority of the chromatographic techniques, the stationary phase is beholden inside a column and the mobile phase is transmitted through pressure or flow with the help of gravity and is called column chromatography. On the other hand, planar chromatography has a piece of paper or a layer of a solid substance coated on another surface, as a stationary phase.

Based on the way of intake and movement of the substance, there are two major techniques. Firstly, there is displacement chromatography in which the mobile phase is strongly retained by the stationary phase, displacing the ingredients of the mix in the column to a different scale. Secondly, the most commonly used and the greatest technique is elution chromatography. In elution chromatography, the ingredients of the mix are transferred inside the column on different speed.

MODERN USES OF CHROMATOGRAPHY

Chromatographic techniques find application both in the chemical analysis and in the isolation and receipt of the separated components. Analytical chromatography finds applications in research, in the development of methods for determining the components that are separated and in quality control. Preparative chromatography applies to the isolation and receipt of small or large quantities of pure component ingredients for a particular use. Chromatography now applies not only to Chemistry, but also to other disciplines such as Biology, Medicine (Diabetes Chromatography Studies), Pharmacy, Biochemistry, Environmental Science, Food Sciences, Agriculture and Toxicology.

Liquid chromatography has applications in all fields of chemical analysis on both analytical and preparative. Firstly, it contributes to the development of the pharmaceutical industry, as antibiotics, steroids, analgesics and vitamins support their research into it. In addition, in the field of Biochemistry, amino acids, proteins, hydrocarbons and lipids and their property are analyzed. Moreover, technical materials, antioxidants and other food ingredients have come from the observation of various liquid chromatography experiments. Furthermore, chemical industries support their research in this form of chromatography. Typical examples in this area, are detergents and dyes. Additionally, the procedural toxicology studies conducted on each new marketed drug and addictive substances (e.g. drugs) are based on liquid chromatography. Environmental studies are also based on this, such as research on pesticides, herbicides, polychlorinated biphenyls and dioxins. Finally, in Clinical Chemistry, drug metabolites and certain hormones in the human body (such as estrogens) are analyzed as it is seated above.

Paper chromatography has become standard practice for the separation of complex mixtures of amino acids, peptides, carbohydrates, steroids, purines and a long list of simple organic compounds. Inorganic ions can also readily be separated on paper.

Thin-layer chromatography has a distinct advantage over paper chromatography because the thin-layer chromatographic plate or sheet is able to withstand strong solvents and color-forming agents.

Gas chromatography is widely used for quantitative and qualitative analysis of mixtures, for the purification of compounds, and for the determination of such thermochemical constants as heats of solution and vaporization, vapor pressure, and activity coefficients. Gas chromatography is also used to monitor industrial processes automatically: gas streams are analyzed periodically and manual or automatic responses are made to counteract undesirable variations. Many routine analyses are performed rapidly in medical and other fields. For example, by the use of only 0.1 cubic centimeter of blood, it is possible to determine the percentages of dissolved oxygen, nitrogen, carbon dioxide and carbon monoxide. Gas chromatography is also useful in the analysis of air pollutants, alcohol in blood, essential oils and food products.

To conclude, nowadays, all the different types of chromatography in general are extremely useful in Science. A huge number of people have been rescued from death, as many medicines have been discovered with the help of chromatography.

A BRIEF HISTORY OF CHROMATOGRAPHY

Chromatography has provided significant contributions to the fields of molecular characterization as well as purification over the last century, thereby being quite unique in its flexibility and scalability. The development of biotherapeutics would have been impossible without chromatography-based purification strategies. Future areas for exploitation are the combination with other separation concepts together with model-based approaches for materials and process design.

The history of chromatography begins during the mid-19th century when a rudimentary version of the technique was used for the separation of plant pigments such as chlorophyll. The first chromatography column was developed by the Russian botanist Mikhail Tswett (1872-1919) who is considered the father of chromatography, in 1901, washed an organic solution of plant pigments through a vertical glass column packed with an adsorptive material. He discovered that the pigments separated into a series of discrete colored bands on the column, divided by regions entirely free of color.

Column chromatography was popularized during the 1930s when the chemists Richard Kuhn and Edgar Lederer successfully used the technique to separate a number of biologically important materials. Since that time, the technique has advanced rapidly and column chromatography is now used widely in many different forms. The column itself has also been refined over the years, according to the type of chromatography, but fulfils the same essential separating function in all forms of column chromatography.

Chromatography is still developing, as many new methods and techniques are still being discovered. Today, chromatography is the best method for separation and analysis of complex mixtures and isolation of susceptible substances. It also has applications on many other sciences such as biology, medicine, pharmacy, environmental science, food sciences and agriculture.

Important dates of the history of chromatography are: (a) first ion exchange resin composition (Adams and Holmes, 1935), (b) development of Liquid-Liquid Chromatography (Martin and Synge, 1941) [Nobel Prize, 1954], (c) Gas Chromatography Development (Martin and James, 1952) and (d) high-performance liquid chromatographic development (last years).

BASIC PRINCIPLES OF CHROMATOGRAPHY

Chromatography is a process by which substances can be separated from their mixtures. There are many types for each of which different materials are used but they work in the same way. The mixture to be analyzed is placed on one end of the material, called the "stationary phase" and is carried to the other end of the material by a fluid, called the "mobile phase". As some substances are more soluble in the fluid, they travel at faster speed, and thus their separation is achieved. To explain it further, as the liquid begins to move towards the solid, some of its molecules are directed toward the surface of the solid material used as a base and temporarily stick there, and then go back to the liquid from which they originated. This particle exchange between the surface of the solid and the liquid is a kind of an adhesive effect known as “adsorption”. That is, the state in which the molecules of a substance are temporarily trapped in another body. Of course, the liquid being analyzed is actually a mixture of quite different liquids, so each molecule is subjected to adsorption in a slightly different way and molecules spend more or less time in the solid or liquid phase. For example, the molecules of a liquid could remain attached to the solid for much longer than the molecules of another fluid of different composition. Another fluid could spend less time on the solid and more on the liquid, so it would go a little faster. Often, the mixture to be analyzed is compared to a glue in order to portray how some molecules stick to the solid more than others. This somewhat complicated procedure explains how fluids travel at different speeds. There are also some standard terms used in chromatography to describe the procedure mentioned above and the most common are:

1. Mobile phase or carrier: solvent
2. Stationary phase or absorbent: stable substance that stays fixed
3. Eluent: fluid entering the mobile phase
4. Eluate: fluid exiting the mobile phase
5. Analyte: mixture whose individual components have to be separated

In order for chromatography to work effectively, it is necessary for the components of the mobile phase to separate out as much as possible as they move past the stationary phase. The stationary phase is the one that stays motionless throughout the whole procedure and allows the sample to move over it. It can be either a solid or a liquid. If it is a solid stationary phase, it is better to have particles of uniform size and shape, preferably spherical. It can also be something with a large surface area, such as a sheet of paper (this is known as paper chromatography), a solid made of finely divided particles, a liquid deposited on the surface of a solid, or some other highly absorbent material. The mobile phase is always a gas or a liquid. It is also important to remember that not everything can be used for any type of chromatography. More specifically, the absorption depends on the polarity of the liquids and the stationary phase. If the stationary phase and the components of the liquid are polar, the flow rate of the fluid is slow and therefore separated from the remainder of the mixture last. Similarly, if both the

stationary phase and the component are non-polar, then the non-polar component comes out last due to slow rate of travel under the influence of mobile phase. So, the stationary phase and mobile phase are always opposite. This means that if the stationary phase is polar, the mobile phase must be non-polar and vice-versa.
Lastly, when chromatography is conducted in a lab, we need to take into account many other factors, like the flow rate or the temperature.

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WORKSHOPS

WS23. PLAYING WITH FAIR AND BIASED COINS

Karol Gryszka
Pedagogical University of Cracow, Poland

ABSTRACT

In this article we provide a description of fair and biased coins which is suitable for all pupils. We discuss the fairness of coins and other random number generating toys, like dice. We also describe how to generate, in some sense, any reasonable distribution, that is to assign equal probability to any number of outcomes. The algorithms whose efficiency is tested will be provided.

1. INTRODUCTION

A coin can be either fair or biased. A fair one has a 50% chance to land on each side: heads or tails. It also means that we exclude the extremely rare yet possible situation where the flipped coin ends on the edge (see [1] for more information about the problem). Biased coin, on the other hand, does not have equal chances for both sides. In particular, one can expect that one side appears more often than the other after repeatedly flipping the coin hundreds or thousands of times.

In this article we provide a simple solution to a well-known problem of the biased coin. Then we use some ideas from binary systems to change any coin into a biased one. The idea turns out to be very simple and what is striking is that it only takes two coin flips, on average, to generate any bias.

We discuss a “coin to dice” problem, that is a way to use the coin to produce dice roll, or more generally, to produce any number from the certain range with equal probability.

To make the article easier to follow, we introduce the necessary definitions and notations whenever they are required instead of including them all in a single section. We also use the following symbols H and T as abbreviations of heads and tails, respectively.

2. HOW TO REMOVE THE BIAS?

Assume that the coin is biased and the probability of receiving H is equal to p. In other words, the value p describes the chance (after multiplication by 100%) of receiving H. Our goal is to somehow balance the coin so that by using a specific set of instructions one can with equal probability obtain either side of the coin.

The beautiful and very simple algorithm by John von Neumann [2] solves the stated problem.

Theorem 2.1. Given any biased coin with the probability $p \in (0,1)$ of receiving heads, the following algorithm produces a fair coin:

1. Flip the coin twice and record both results.
2. If the resulting sides differ (HT or TH), assign the first result with the final outcome.
3. If the resulting sides coincide (HH or TT), go back to Step 1.

The algorithm works simply because the probability of receiving two different sides is equal and does not depend on the order of coin sides.

Example 2.2. John flips a biased coin twice and receives TT. According to the algorithm, he must repeat the process again. Then he flips HH. He repeats again and obtains HT. He stops and call the result heads.
Let us explain what does it mean to produce a fair coin. Technically, we use two coin flips to obtain a single result. So while in practice we have no physical form of the fair coin that flipped once gives the desired result, in theory, and using a pre-determined set of flips we can assign to the entire process of flips one of the sides.
The dice or any other number-generating toll can also be partially biased. Actually, we never know whether the dice is balanced in such a way each number can be obtained with equal probability. Due to small imperfections, the number 1 can, for instance, be 0.001% more common than 6. The von Neumann algorithm can be generalized to resolve the issue. The brief description for the case of 6-sided dice is as follows: roll the dice 6 times until 6 different numbers appear. Then call the first received number the final outcome. The problem with such an approach is massive – it is very impractical to do without any computer as it requires, for 6-sided dice case, at least 330 dice rolls on average!

3. ADDING THE BIAS TO A FAIR COIN

We now discuss the converse problem. In order to do that we need a necessary introduction to the binary system.
Binary numbers are very similar in their nature to decimal numbers. The term “decimal” means the base in which the number is represented is equal to 10. Look at the following example:

$$1514 = 1 \cdot 10^3 + 5 \cdot 10^2 + 1 \cdot 10^1 + 4 \cdot 10^0.$$

Each number in decimal, or base 10 system, can be written as a sum of powers of 10, each power multiplied by one of the allowed digits:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

The powers in that example are non-negative, meaning that we only consider integer numbers. That can be extended to fractions by using the obvious assignment:

$$3.14 = 3 \cdot 10^0 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2}.$$

The binary, or base 2 system, uses only two digits: 0 and 1. The representation of any number follows the similar rule, but this time we have to use powers of 2:

$$101_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0.$$

Here and later if it is necessary, we include subscript to the number to indicate which base the number is written in. We use 2 for the binary and 10 for the decimal system. The same rule of representation applies to fractions:

$$0.011_2 = 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3}.$$

Note that the numbers on the right-hand side of the last two equations can be interpreted as decimal numbers, so that

$$101_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 4 + 0 + 1 = 5_{10}$$

as the number in the decimal system, and similarly:

$$0.011_2 = 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375_{10}.$$

This can be easily applied to any number in the binary system to find its value in the decimal system. The actual problem is how to change the number in decimal to binary. We will provide the algorithm for fractions only and we refer to [3] for integers.
Let p be any number in the range $(0,1)$. Our goal is to find the binary representation of that number. Then

$$p = 0.b_1 b_2 b_3 b_4 \ldots,$$

where the numbers b_i for each i are found using the following scheme:

Multiply the number p by 2. If the result is smaller than 1, set $b_1 = 0$, else if the result is greater than or equal to 1, set $b_1 = 1$ and subtract 1 from the value of 2p. In either case, call the resulting value on p by p_1.
Multiply the number p_1 by 2. If the result is smaller than 1, set $b_2 = 0$, else if the result is greater than or equal to 1, set $b_2 = 1$ and subtract 1 from $2p_1$. In either case, call the resulting value on p_1 by p_2.
Repeat the procedure as long as $p_k = 0$ for some k or when the desired accuracy of representation is obtained.

Example 3.1. If $p = \frac{1}{3}$, then:

- $2p = \frac{2}{3} < 1$, so $b_1 = 0$ and $p_1 = \frac{2}{3}$,
- $2p_1 = \frac{4}{3} \geq 1$, so $b_2 = 1$ and $p_2 = \frac{4}{3} - 1 = \frac{1}{3}$,
- $2p_2 = \frac{2}{3} < 1$, so $b_3 = 0$ and $p_3 = \frac{2}{3}$,
- $2p_3 = \frac{4}{3} \geq 1$, so $b_4 = 1$ and $p_4 = \frac{4}{3} - 1 = \frac{1}{3}$,
- and so on.

From that, we can clearly see that we obtain an alternating sequence of 0's and 1's, hence

$$\frac{1}{3} = 0.0101010101\ldots_2 = 0.(01)_2,$$

where the bracket notation means the number inside the bracket repeats infinitely.
It is clear that using the above algorithm any number from the introduced range can be expressed in binary system. We can now state the algorithm that generates the bias from a fair coin.

Theorem 3.2. Given a fair coin, let

$$p = 0.b_1b_2b_3b_4\ldots$$

be the binary representation of the number $p \in (0,1)$ that describes the probability of biased coin to flip heads. The following algorithm generates the biased coin flip:

- Flip a coin until it records heads for the first time. Assume that this happens in the n-th flip.
- If $b_n = 1$, then we call the final result heads.
- If $b_n = 0$, then we call the final result tails.

Remark 3.3. The algorithm described in Theorem 3.2. requires on average 2 coin flips.
We suggest the following experiment to be conducted by the Reader.

Problem 3.4. Find the binary representation of $p = 1/4$. Flip biased coin with probability p of receiving heads at least 50 times and calculate the number of heads and tails received according to Theorem's 3.2 algorithm. Calculate the total and the average number of flips required to complete the experiment.

4. GENERATING A DICE ROLL WITH COIN FLIPS

Let us now assume that we have a fair coin in use. We can always do that due to Theorem 2.1 – any biased coin can be turned back into a fair one using von Neumann's approach. Having a fair coin in our hands, we want to obtain one number from the set

$$\{1, 2, 3, 4, 5, 6\},$$

that is we want to simulate a dice using a given coin. Let us first provide some simple definitions. Assume that we are conducting an experiment, name it X, with N possible outcomes.

Definition 4.1. We say that X with N outcomes has uniform distribution if the probability of receiving any outcome is equal to $1/_N$, which we denote in the following way:

$$P(X = i) = \frac{1}{N}, \qquad i = 1, \dots, N.$$

Here $X = i$ means that experiment X ends at the result i. We say that X has the uniform N-point distribution.
One can also think of probabilities as some weights attached to each number. Each number is then selected according to their weight. The weights can, in general, be different from one another, however to avoid difficulty and to focus the attention on one topic we skip that approach.

Example 4.2. Let X be the dice roll. Then $N = 6$ and

$$P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = \frac{1}{6}.$$

We now introduce the idea of the distribution function. Let B be any subset of the set $\{1, 2, \dots, N\}$. We can now define a function that describes the probability for X to receive one of the outcomes in B. That probability is just the sum of each individual outcomes in B.

Definition 4.3. The function P_X given by

$$P_X(B) \coloneqq P(X \in B) = \frac{|B|}{N}$$

and defined for all B – subsets of $\{1, 2, \dots, N\}$ is called a distribution function of X. Here $|B|$ denotes the number of elements in the set B.

Example 4.4. The X be the dice roll. Set

$$B_1 = \{1\}, B_2 = \{1,2,3\}, B_3 = \{2,4,6\}, B_4 = \{2,3,4,6\}.$$

Then

$$P(X \in B_1) = \frac{1}{6}, P(X \in B_2) = \frac{1}{2}, P(X \in B_3) = \frac{1}{2}, P(X \in B_4) = \frac{2}{3}.$$

With the above notation, we can now formulate the following.

Theorem 4.5. Let X be the dice roll experiment, that is let X be the uniform 6-point distribution. Then there exists an algorithm that generates X using fair coin flips.

Proof. The proof is based on the following. We flip the coin three times and compose a 3 letter word (like HHH or THT) from the results. Then we create the following assignment:

$$\begin{aligned}
&HHH \to 1,\\
&HHT \to 2,\\
&HTH \to 3,\\
&HTT \to 4,\\
&THH \to 5,\\
&THT \to 6,\\
&TTH \to repeat,\\
&TTT \to repeat.
\end{aligned}$$

Here, “repeat” means we have to flip the coin three times again. We repeat that process until one of the numbers is selected.
Let us check that $P(X = 1) = 1/6$. We decompose the calculation as follows. The event “$X = 1$” can happen in the first 3-flip. Let Y denote the experiment of throwing the coin 3 times. Then

$$P(Y = HHH) = \frac{1}{8}.$$

The chance to repeat is

$$P(Y = TTH \ or \ Y = TTT) = \frac{1}{4}.$$

Then, if the algorithm has to be repeated, the repetition always happens with the above probability and the chance of completing the algorithm after each repetition is always $1/8$. Gathering all possibilities, that is "$Y = HHH$" received in 1st 3-flip, 2nd 3-flip, 3rd 3-flip etc. we obtain the series

$$\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4^2}\cdot\frac{1}{8} + \frac{1}{4^3}\cdot\frac{1}{8} + \frac{1}{4^4}\cdot\frac{1}{8} + \cdots = \frac{1}{6},$$

where the value of the sum follows from the sum of the geometric series formula.
Using the same principle we check that

$$P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = \frac{1}{6},$$

as required. Q.E.D.

The coin can be used to generate any uniform N-point distribution. We skip the proof of that fact.

Theorem 4.6. Let X be any uniform N-point distribution. Then there exists an algorithm that generates X using coin flips.
We shall now explain how to change a coin into dice using a slightly different approach.

Example 4.7. Flip a coin once and use the following assignment:

$$H \to 1,2,3,$$
$$T \to 4,5,6.$$

This means that when you receive H, the next step of the algorithm will determine if the final outcome is 1, 2 or 3. That final step is to flip the coin twice and use the following assignment:

$$\begin{array}{ll} case\ H & case\ T \\ HH \to 1, & HH \to 4, \\ HT \to 2, & HT \to 5, \\ TH \to 3, & TH \to 6, \\ TT \to repeat, & TT \to repeat. \end{array}$$

If in the final step we receive HH, then we call the final result 1. If it was TT, then we flip the coin twice again and if in that attempt we receive TH, we call the final result 3.

Problem 4.8. Which algorithm is better? Check the efficiency of both algorithms presented in Example 4.7 and Theorem 4.5.
The algorithm used in the proof of Theorem 4.5 can be generalized to generate uniform 7-point distribution. The generalization is left to the Reader.

Problem 4.9. Construct a coin-based algorithm that would choose one number from the set $\{1,2,3,4,5,6,7\}$ with equal chances for each number.
This can be extended even further by increasing the number of flips to generate any uniform N-point distribution. We propose the following problem.

Problem 4.10. Construct a coin-based algorithm that would choose one number from the set $\{1,2,3,4,5,6,7,8,9\}$ with equal chances for each number (hint: flip the coin 4 times and create a proper assignment).
We finish this section with the following problem. Note that we assume that the Reader solved Problems 4.9 and 4.10 before.

Problem 4.11. How good the algorithms created in the last two problems are? Check the number of flips (both total and the average) and explain why the average in Problem 4.10 is significantly larger than the one in Problem 4.9. Can you also explain why 3 flips will not suffice for any assignment in Problem 4.10? Up to which value of N exactly 4 flips are sufficient to create at least one assignment?

5. FINAL REMARKS

Using just a coin flip we can do almost anything that requires selecting an outcome from the given set. The algorithms presented in this paper provided a very basic approach to the topic. In fact, one can do much more than just selecting something from the given set. The ideas can be generalized not only to those distributions that have equal chances but also to those whose chances differ. Moreover, one can even describe a coin-based algorithm that would select one number from the set of all(!) natural numbers with a proper distribution of weights among these numbers.

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